A Logical Expression of Reasoning

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Abstract. A nonmonotonic logic, the Logic of Plausible Reasoning, LPR, capable of coping with the demands of what we call *complex reasoning*, is introduced. It is argued that creative complex reasoning is the way of reasoning required in many instances of scientific thought, professional practice and common life decision taking. For managing the simultaneous consideration of multiple scenarios inherent in these activities, two new modalities, weak and strong plausibility, are introduced as part of the Logic of Plausible Deduction, LPD, a deductive logic specially designed to serve as the monotonic support for LPR. Axiomatics and semantics for LPD, together with a completeness proof, are provided. Once LPD has been given, LPR may be defined via a concept of extension over LPD. Although the construction of LPR extensions is first presented in standard style, for the sake of comparison with existing nonmonotonic formalisms, alternative more elegant and intuitive ways for constructing nonmonotonic LPR extensions are also given and proofs of their equivalence are presented.

Keywords: ampliative reasoning, complex reasoning, nonmonotonic logic, default logic, epistemic modalities, paraconsistency

1. Introduction

1.1. Relationship of logic and reasoning

Logic provides reasoning[1] with a way of expression and a norm. Deductive logic does it to a very particular kind of reasoning, the most

[1] It must be made crystal clear that we associate no psychologist connotations to the term "reasoning". Following a methodological tradition inaugurated by Frege, which is in the root of the so called *linguistic turn*, even to words like "thought" it is given a mathematical abstract sense. As Wittgenstein states it — "It is misleading then to talk of thinking as of a 'mental activity'. We may say that thinking is essentially the activity of operating with signs" (cf. [26], p. 6). So is "reasoning".

praised one as far as philosophy, mathematics and sciences (hard sciences) are concerned. Deductive reasoning has the nice property of being conservative with respect to truth. It is able (or at least it is intended to be able) to carry truth on all the way from premises, once it is there, to conclusion. Being truthfulness such a precious good, a reasoning that does not assure its integrity has not deserved respect and serious concern from the tradition. That is why for so long, at least until Hume [12], but even after him (deductive) logic and correct reasoning, or even "rationality", has been taken as synonymous.

However, real life reasoning, meaning a wide range of forms of inference, covering from common sense to scientific reasoning, passing through reasoning required for technical professional practice, such as in law, economics, medicine and so on, is a complex, plural, multifaceted matter. It is in such a broad spectrum that it is considered in the present paper. Once reasoning is considered in those general terms its relation to logic becomes much more problematic and much less strict and clear. Questions then arise even whether it is at all possible to establish insightful connections between logic and reasoning. And if so, what those connections are? Or, better stating, what role is logic still able to play as a tool for clarify and express reasoning? We intend to answer to those questions in an effective way by presenting a logic able to express reasoning understood in this large spectrum.

1.2. Logic beyond analytical reasoning

Of course, in order to do so, we have to consider logic in a *lato sensu* as well, for if it is understood as deductive logic it has not too much to say about the many forms of non deductive, superdeductive, inductive, estimative, imprecise, hypothetical, evidential, plausible varieties of reasoning practice. So, we take the word "logic" here as denoting a certain class of mathematical systems for which classical logic provides a reference and a model, but it is not its sole sample. What is kept of classical logic is its mathematical style and a stockpile of tools and results that are still valid or can be adapted to this extended formulation. Within this spirit, the problem we face could be better stated as the task of designing logics able to analyze, annotate and express some relevant features of certain kinds of non deductive reasoning.

A similar project in relation to the justification of induction in the construction of scientific theories has been worked out by a variety of authors including Carl Hempel, Rudolf Carnap and Patrick Suppes [11, 5, 25] among others. However, the preferred approach to the induction problem has been a quantitative one, based on variations of probability theory. More recently, in the field of artificial intelligence, qualitative approaches, with more resemblance to the discipline of logic, the so called nonmonotonic logics, have been proposed to treat a form of non deductive reasoning, sometimes called "common sense reasoning". We took this problem in AI as our initial motivation and starting point, but we have evolved, along a ten years project, to consider the problem of providing a mathematical qualitative modeling for non deductive reasoning taken in greater generality, including some complex aspects not usually treated in the AI literature, such as what Hempel has called "the problem of inductive inconsistencies", and the problem of reasoning by considering the plausibility of multiple alternative scenarios. What is presented in this paper is the result of this effort in the form of a logic that we propose as a contribution to the understanding of non deductive reasoning in many of its instances, including those relevant for science, such as reasoning in the presence of competing hypothesis or theories, and inference from uncertain or quasi universal conjectures. Furthermore, we believe that the logic being proposed is relevant for philosophy of language (concerning the pragmatics of language); practical philosophy (concerning ethical judgment, rationality and decision taking for action); economic analysis (taking into account different plausible scenarios); and so on.

1.3. The complexities of real reasoning

The most preeminent feature when real, practical reasoning, let's call it this way, comes into play is its way of inference, which is not, strictly speaking, truth preserving. Being truth something that one is much in doubt to possess from the beginning, when practical, or even scientific matters are in order, its strict preservation is not an absolute value for this particular kind of reasoning. Besides, in order to preserve truth, the inference power is restricted to such an extent that the logic would become useless in many practical applications. What it is reasonable to preserve, instead of certainty, is some sort of "degree of confidence", let's say so, less or more high, depending on the nature of the problem and the requires on accuracy in treating it. This "degree of confidence" is designated by many names, such as probable, expectable, reasonable, and the like. Here we adopted the adjective *plausible* to play this role. Accordingly, it is said that the logic proposed in the body of the paper is capable to express *plausible reasoning.*

However, in spite of inferential enlargement be an essential feature of practical or scientific reasoning, it is not the only feature to provoke a departure of reasoning from traditional logic. There is another equally relevant but often neglected aspect. It consists of a phenomenon that emerges while considering the following question — why so many times the reasoning seems to be impeccably logical although leading to very stupid conclusions? The standard answer is — because one starts from erroneous, may be stupid premises. But this does not tell the whole story. Sometimes this may happen even when the premises also seem very reasonable. Our diagnosis for this problem is lack of imagination. It is the way of preference to prejudice, fanaticism, dogmatism, patriotism and other isms of the like. It is not solved by the mere change of premises, which may just switch from one ism to another. The rather radical, though appropriate, solution is to take into consideration all looking reasonable premises — even if they, eventually, as they often do, contradict each other — and then to reason them out altogether. What emerges of this reasoning melting pot is an open mind consideration of different plausible scenarios. This is a wise, imaginative and effective way of reasoning, commonly used in technical, professional and scientific practice, but hardly, if ever, treated through the use of logic. The logic presented here accounts for the expression of multiple, imaginative reasoning, which we call *complex reasoning.*

1.4. Scientific reasoning is ampliative complex reasoning

Perhaps, the greatest contribution of Hume to clarify the matters about the acquisition of knowledge from observation and experimentation was precisely to make the decisive remark about this epistemological fact: scientific reasoning is not supported by logic, or, better stating, it is not analytical reasoning. But, if scientific reasoning is not truth preserving deductive reasoning, what type of reasoning is it? This was the problem raised by Hume, and it is still open nowadays: the problem of characterizing ampliative reasoning and of distinguishing it from fallacies or even plain irrationality. “Why is rational to take non conservative conclusions, from which so much of our knowledge depends on?”; “is it rational to take these conclusions?”; “when is rational to take them and when is not?” Those are questions open to answer.

Well, it is rational to take those conclusions in certain conditions. The task of the ones that have dedicated themselves to this problem is precisely to expose what constitutes these conditions. This is our task here, but, before we go on, the usage we do of some terms must be clarified. We call *conservative inference* deductive reasoning in general. This lies in opposition to the term *ampliative inference* which denotes all kind of non deductive inferences that are not just fallacies. Many authors, in particular those concerned with the justification of induction as part of the scientific method, consider all kinds of ampliative inference as induction. So the universe of inference is divided in two mutually excluding parts: deductive and inductive, being inductive inferences thus simply defined as those inferences which are not deductive. For the sake of clarity and more specificity, here we call those inferences which are not deductive as forming the class of *ampliative inferences*, and reserve the term induction to a special subclass of it. In what this subclass consists of and what distinguishes it from other ampliative inferences? The authors who discuss the role of induction in science agree that an important feature of induction is that it is a kind of inference in which the conclusion is a more primitive, or general, statement than the data from which it is concluded. We call this an *ascendant inference*, an inference going from the particular to the general or to more general regularities anyway. But there are also inferences that are non deductive and *descendent*, in the sense that they go from general statements to more particular conclusions, as deduction does, being non deductive, because departing from generalizations which cannot be just taken as certainties, and creative, in opposition to the conservativeness of plain deduction. This stems from the fact that the principles and generalizations they depart from are not precise statements but statements that may admit exceptions, working in a scenario of incomplete knowledge. So they also are ampliative inferences, but not ascending ones, and so, we do not consider really appropriate to call them “inductions”.

A relevant question is: does this kind of inference plays a role for scientific reasoning, as the other two certainly do? It certainly plays a role, a preeminent one, for complex reasoning, but do they do the same for scientific investigation?

We answer this question in the affirmative, and the present paper, as far as it claims to present a contribution for systematization of reasoning which is relevant for science, is in a great deal a consequence of this question being answered this way. In so doing, we do not stand alone. Newton da Costa and contributors, for instance, have suggested in [6, 7] the concept of pragmatic truth as playing a role in scientific practice; to that concept naturally corresponds a way of reasoning which is ampliative. To be convinced on that point we have to distinguish between *scientific investigation* — whose outcome may be a theory, or theories — from *scientific application*, the use of a theory taken as established in its principles. Moreover, we must have in mind that the spectrum of theories in what is nowadays accepted as science, and their corresponding investigative activity, which can be called *scientific reasoning*, is a large and diverse one, a fact which is frequently neglected in discussions about science and scientific reasoning. At one end of the spectrum we find the canonical analytic reasoning of deductive sciences. Those are the “hard” sciences, which rely primarily, if not completely, on mathematically axiomatized theories. Frequently, the analysis of scientific theories in the literature of philosophy of science restricts itself to the consideration of this kind of theory. However, within the spectrum of real science, they make more the exception than the rule, although they undoubtedly have the strong appeal of serving as a paradigm, a utopia every science should strive to achieve. This is the case of the so called social sciences, such as sociology and economy, but it is also the case of practical sciences, such as engineering and medicine, not to mention law. It is as well the case of sciences in its early stages of construction, such is cognitive science nowadays. So not all of scientific reasoning, even descending reasoning, is deductive.

To the question on the rationality of performing ampliative inference, we certainly answer in the affirmative: it is rational to do it in some circumstances. What is not rational, and this is well established since Hume's remark, is to take the conclusions so reached as certainties, i.e., to given them the same status of deductive conclusions. We avoid this misuse by taking them as just *plausibilities*, to be distinguished from deductive conclusions by marking them with epistemic modality symbols, the question mark "?", standing for *weak plausibility*, and the exclamation mark "!" for *strong* (or *strict*) *plausibility.*

1.5. The role of exceptions

Our experience in dealing with ampliative reasoning expressed in terms of rules subject to exceptions has teach us that the whole logistic of the processes is very sensitive to the treatment it is given to exceptions. We realized that the relationship of exceptions with the rule they belong to is subtle, if not plainly tricky, and plays a decisive role in how inferences should be done. Exceptions convey a sort of "meta-knowledge" about the usage of rules. Furthermore, there is something paradoxical in the relation of exceptions with the rules they refer. As a Minister of the Brazilian Supreme Court once stated — "In face of exception, the rule applies by not applying itself." There is a scent of Russell's paradox in the air. In order to take care of such potential paradoxes it must be realized that exceptions induce a hierarchy among rules. The exceptions to a rule must be derived independently of it. A rule stays in a higher position in relation to any other (rule) relevant to the derivation of its exception. A rule cannot interfere (either to confirm or deny) with the derivation of its own exception. Only after the issue about whether the exception is the case (or not) the activation is settled and the rule may come into play (by applying or un-applying, whether the case may be). There is, thus, a certain parallel with typed set theory here: the elements of a set must be given beforehand its construction, a set cannot be a member of itself nor can two sets be a member of each other reciprocally.

This issue plays a central role in our approach. It is important to characterize defective theories through the detection of cycles: rules relevant to the derivation of each other exception or to their own exception. A cyclic theory may have more than one or even none extension (in some fortuitous cases, they might even have only one extension as the normal, acyclic theories always do). But, even more important, the hierarchy induced by the exceptions is relevant to the generation of extensions, and, consequently, to the determination of plausible scenarios. In the frequent event that the application of a rule leads to the derivation of an exception to another rule, only a plausible scenario emerges from the interplay of these rules: the one generated by the first rule (the second rule being precluded by the derivation of its exception). The alternative scenario where the application of the second rule would block the derivation of its exception violates this hierarchy, it is so prevented in LPR. Following [18] it is said that LPR complies with the *exceptions-first criterion*.

This is an important feature distinguishing LPR from traditional approaches to nonmonotonic reasoning in Artificial Intelligence, namely Circumscription [16] and Default Logic [23]. As a matter of fact, to this date, the authors do not know any formalism to nonmonotonic reasoning which gives to the hierarchy among rules induced by their exceptions the recognition and the importance it deserves.

1.6. Plan of the paper

In this paper two intertwined logics are presented, the *Logic of Plausible Deduction* — LPD — and the *Logic of Plausible Reasoning* — LPR. LPD is a deductive monotonic logic which formalizes reasoning with multiple scenarios. An earlier version of this logic was presented in [21] and [4]. In [19] it was presented the Inconsistent Default Logic (IDL), from which LPR is, in a sense, its successor. In [20] there is a monotonic basis for IDL, the Logic of Epistemic Inconsistency, from which LPD is its successor. It generalizes the modal logic S5, since it works with two collections of worlds instead of only one as in Kripke possible worlds semantics [14]. The first collection comprehends the possible worlds, and the second, a subcollection of the first, encompasses the plausible worlds. This allows for the introduction of two new modalities besides the traditional alethic ones for *possibility* ($\Diamond$) and *necessity* ($\Box$). The newly introduced modalities denote the epistemic status of the plausible statements, distinguishing them from the ones taken as certainties. $P!$, *strong* or *strict plausibility*, means that the assertion P holds in all plausible scenarios, whereas $P?$, *weak plausibility* or simply *plausibility*, means that P holds in at least one plausible scenario. Since the collection of plausible worlds is a subset of the possible ones, plausibility is stronger than possibility and the expected hierarchy holds in LPD: $\Box P$ entails $P!$, which entails $P?$, which entails $\Diamond P$. In sections 2 and 3, it is presented respectively a semantics and an axiomatics for LPD. Meanwhile, the *Logic of Plausible Reasoning*, LPR, is presented here as our proposed solution to the problem of expressing complex reasoning. From the premises comprehending certain and inconclusive knowledge, it constructs the alternative plausible scenarios. Once conjectural, plausible but less than conclusive, knowledge is represented, the emergence of alternative scenarios is a natural consequence. The plausible scenarios are determined using the notion of *extension*, as it is defined in section 4. Then, the alternative scenarios are reasoned out using the logic specially designed for this purpose, LPD. The theorems, assigned to the proper modalities, are, then, inferred. $\Box P$ for those holding in all scenarios; $P!$ for those holding in all plausible scenarios; $P?$ for those holding in some plausible scenarios and $\Diamond P$ for those holding in some scenarios. LPR is presented in section 4. Alternative ways of determining scenarios are presented in section 5; we hope it helps to clear up the concepts involved. Finally, in section 6 we present our conclusions.

2. A Semantics for Plausible Deduction

In this section a semantics for the Logic of Plausible Deduction is provided. This is done by first introducing the concepts of LPD-*structure* and LPD- *interpretation*. An LPD-structure consists of two collections of *classical structures*, or *worlds*, over a same *universe* or *domain*. The first collection is called the set of *possible worlds*, and the second, which is a non empty subset of the first, is the set of *plausible worlds*. An LPD-interpretation, associated with a given LPD-structure, picks up a world and establishes an assignment of variables into the common domain. Associated to a given LPD-interpretation Θ, two functions are defined: the first one, Θ_{D}, called *the denotation defined by* Θ, assigns an object of the universe of the interpretation Θ to each given term; the second one, Θ_{E}, assigns a truth value to each given formula. The function Θ_{E} is called *the evaluation function defined by* Θ. Finally, from the functions of the form Θ_{E}, it is defined, for each LPD-structure H, the LPD-valuation H_{V}. The truth values of LPD are 1 and 0, whereon 1 means *true* and 0 means *false*.

2.1 Definition. A language for LPD is a first order language, as it is usually defined in standard textbooks, such as [1, 8, 9, 13, 17, 24], adopting "$\rightarrow$", "$\neg$", "$\Box$" and "!" as primitive connectives, and "$\forall$" as the sole primitive quantifier.

2.2 Notation. From now on, unless declared otherwise, the following conventions are adopted, related to the syntactic variables given below, followed or not by primes and/or subscripts:
- L is a language for LPD;
- x, y, z are variables in any language for LPD;
- t, u are terms in L;
- P, Q, R, S are formulas of L;
- p, q, r are atomic sentences of L, whereon distinct letters represent distinct sentences;[2]
- Γ, Φ are collections of formulas of L;
- Δ is a non empty set.

An LPD-*structure for* L provides a non empty set called *universe*, and, for each possible world, meanings for the constants, functions and predicate signs in L into this universe. These meanings don't vary, along the possible worlds of the structure, for the constants and functions signs in L, but can vary, along these possible worlds, for the predicate signs.

2.3 Definition. A *world* w *over* Δ *for* L is a function satisfying the following conditions:
- if c is a constant in L, $w(c) \in \Delta$;
- if f is an n-ary function sign in L, $w(f)$ is a function from Δ^n to Δ;
- if p is an n-ary predicate sign in L, $w(p)$ is a subset of Δ^n.

A collection W of worlds over Δ for L is said *rigid* if the following clause is fulfilled:
- for each $w, w' \in W$ and for each $\mathcal{S}$, if $\mathcal{S}$ is a constant or a function sign in L, $w(\mathcal{S}) = w'(\mathcal{S})$.

2.4 Definition. An LPD-*structure for* L is a triple $H = \langle \Delta, W, W' \rangle$, whereon Δ is called the universe of H, and W, W' are non empty rigid collections of worlds over Δ for L, such that $W' \subseteq W$, whose elements are called respectively *possible worlds* (*in* W) and *plausible worlds* (*in* W') of H.

[2] See definition 2.14, where it is specified what we mean by "sentence". Although the notation specified in this item was formulated now, it will be used only in section 4.

An LPD-*interpretation for* L, besides providing a universe and meanings for constants, functions and predicate signs in L into its universe, as it is already done by its internal structure, gives a fixed world and meanings for all variables in L. That is essential for providing meanings for all terms and formulas in L, as a first step for specifying a semantics for LPD.

2.5 Definition. An LPD-*interpretation for* L is a quintuple $\Theta = \langle \Delta, W, W', w, s\rangle$, whereon $H = \langle \Delta, W, W'\rangle$ is an LPD-structure for L, $w \in W$ and s is a function from the set of all variables in L to Δ, also called a Δ-*assignment* (*for variables*). It is said, in this case, that Θ is an LPD-*interpretation for* L *over* (the LPD-structure) H (*for* L).

2.6 Definition. If s is a Δ-assignment for variables and $d \in \Delta$, then $s(x|d)$ is the Δ-assignment for variables defined below:

- $s(x|d)(y) = \begin{cases} s(y), & \text{if } y \neq x; \\ d, & \text{if } y = x. \end{cases}$

2.7 Definition. If $\Theta = \langle \Delta, W, W', w, s\rangle$ is an LPD-interpretation for L, $d \in \Delta$ and $w' \in W$, then $\Theta(x|d)$ and $\Theta(w|w')$ are the LPD-interpretations for L specified below:

- $\Theta(x|d) \rightleftharpoons \langle \Delta, W, W', w, s(x|d)\rangle$;
- $\Theta(w|w') \rightleftharpoons \langle \Delta, W, W', w', s\rangle$.

2.8 Definition. Given an LPD-interpretation $\Theta = \langle \Delta, W, W', w, s\rangle$ for L, the following clauses specify the functions Θ_{D} and Θ_{E}:

- Θ_{D} is a function from the collection of terms in L to Δ, called the *denotation for* L *defined by* Θ;
- Θ_{E} is a function from L to $\{0,1\}$, called the *evaluation for* L *defined by* Θ;
- $\Theta_{\mathrm{D}}(x) = s(x)$;
- if c is a constant in L, $\Theta_{\mathrm{D}}(c) = w(c)$;
- if f is an n-ary function sign in L, then $\Theta_{\mathrm{D}}(f(t_1, \ldots, t_n)) = w(f)(\Theta_{\mathrm{D}}(t_1), \ldots, \Theta_{\mathrm{D}}(t_n))$;
- if p is an n-ary predicate sign in L, then $\Theta_{\mathrm{E}}(p(t_1, \ldots, t_n)) = 1$ iff $\langle \Theta_{\mathrm{D}}(t_1), \ldots, \Theta_{\mathrm{D}}(t_n)\rangle \in w(p)$;
- $\Theta_{\mathrm{E}}(\neg P) = 1$ iff $\Theta_{\mathrm{E}}(P) = 0$;
- $\Theta_{\mathrm{E}}(P \to Q) = 1$ iff $\Theta_{\mathrm{E}}(P) = 0$ or $\Theta_{\mathrm{E}}(Q) = 1$;
- $\Theta_{\mathrm{E}}(\forall x\, P) = \min\{\, \Theta(x|d)_{\mathrm{E}}(P) \mid d \in \Delta \,\}$;
- $\Theta_{\mathrm{E}}(\Box P) = \min\{\, \Theta(w|w')_{\mathrm{E}}(P) \mid w' \in W \,\}$;
- $\Theta_{\mathrm{E}}(P!) = \min\{\, \Theta(w|w')_{\mathrm{E}}(P) \mid w' \in W' \,\}$.

2.9 Definition. Each LPD-structure H for L specifies a function from L to $\{0, 1\}$, denoted by H_{V}, called the LPD-*valuation for* L *defined by* H:

- $H_{\mathrm{V}}(P) = \min\{\, \Theta_E(P) \mid \Theta \text{ is an LPD-interpretation for } L \text{ over } H \,\}$.

2.10 Definition. Let H be an LPD-structure for L.
- H *satisfies* $P \rightleftharpoons H_V(P) = 1$;
- H *satisfies* $\Gamma \rightleftharpoons H$ satisfies each formula of Γ;
- P is LPD-*satisfiable* $\rightleftharpoons$ there is an LPD-structure for L which satisfies P;
- P is LPD-*unsatisfiable* $\rightleftharpoons$ no LPD-structure for L satisfies P;
- P is LPD-*valid* $\rightleftharpoons$ P is satisfied by each LPD-structure for L.

In an analogous way it is defined when a *collection of formulas of* L is LPD-*satisfiable* or LPD-*unsatisfiable*.

2.11 Definition. We say that P is an LPD-*semantical consequence* of Γ if each LPD-structure for L satisfying Γ also satisfies P. When it happens, it is also noted by $\Gamma \models_{\overline{\text{LPD}}} P$.

2.12 Definition. A *world* w *over* Δ *for* L is said *to satisfy* P if the LPD-structure $H = \langle \Delta, \{w\}, \{w\} \rangle$ for L satisfies P. w is said *to satisfy a collection of formulas of* L if it satisfies each formula of this collection.

The definitions above characterize the semantics of LPD as an *open logic*[3]. In an open logic, the rules involving universal quantification (either for variables or worlds) like generalization and necessity are sound while only a restricted form of the deduction theorem holds (see next section). Actually, in these logics, if an implication follows from a collection of formulas, then its antecedent logically implies the consequent under this collection of formulas, but not the other way round. Thus we have the following relation:

2.13 Theorem.
- $\Gamma \models_{\overline{\text{LPD}}} P \to Q$ implies that $\Gamma, P \models_{\overline{\text{LPD}}} Q$,
  but $\Gamma, P \models_{\overline{\text{LPD}}} Q$ does not always imply that $\Gamma \models_{\overline{\text{LPD}}} P \to Q$.

2.14 Definition. An *occurrence of a variable* x is said *bound in* P if it occurs inside a subformula of P of the form $\forall x\, Q$, otherwise this occurrence is said *free in* P. A *variable* is said *free in a formula* if it has at least a free occurrence in this formula. A *sentence* is a formula which does not contain any free variable.

2.15 Definition. Let P_0 be a sentence of L chosen arbitrarily. The following abbreviations are adopted:
- $\top \rightleftharpoons P_0 \to P_0$;
- $\bot \rightleftharpoons \neg(P_0 \to P_0)$;
- $P \wedge Q \rightleftharpoons \neg(P \to \neg Q)$;
- $P \vee Q \rightleftharpoons \neg P \to Q$;
- $P \leftrightarrow Q \rightleftharpoons (P \to Q) \wedge (Q \to P)$;
- $\exists x\, P \rightleftharpoons \neg \forall x\, \neg P$;

[3] The other option would be to define LPD-valuations based on LPD-interpretations; this would make the semantics of LPD a *closed logic*. A general study of open logics and some notes about differences between open and closed logics is done in [2, 3].

- $\Diamond P \rightleftharpoons \neg\Box\neg P$;
- $P? \rightleftharpoons \neg((\neg P)!)$.

2.16 Theorem. Given an LPD-interpretation $\Theta = \langle \Delta, W, W', w, s\rangle$ for L, the semantics in LPD for the defined connectives and quantifiers is given below:

- $\Theta_{\mathrm{E}}(\top) = 1$;
- $\Theta_{\mathrm{E}}(\bot) = 0$;
- $\Theta_{\mathrm{E}}(P \wedge Q) = \min\{\Theta_{\mathrm{E}}(P), \Theta_{\mathrm{E}}(Q)\}$;
- $\Theta_{\mathrm{E}}(P \vee Q) = \max\{\Theta_{\mathrm{E}}(P), \Theta_{\mathrm{E}}(Q)\}$;
- $\Theta_{\mathrm{E}}(P \leftrightarrow Q) = 1$ iff $\Theta_{\mathrm{E}}(P) = \Theta_{\mathrm{E}}(Q)$;
- $\Theta_{\mathrm{E}}(\exists x\, P) = \max\{\Theta(x|d)_{\mathrm{E}}(P) \mid d \in \Delta\}$;
- $\Theta_{\mathrm{E}}(\Diamond P) = \max\{\Theta(w|w')_{\mathrm{E}}(P) \mid w' \in W\}$;
- $\Theta_{\mathrm{E}}(P?) = \max\{\Theta(w|w')_{\mathrm{E}}(P) \mid w' \in W'\}$.

As it is intended there is an epistemic hierarchy among the formulas of LPD as follows:

2.17 Theorem.

- $P \models_{\overline{\mathrm{LPD}}} \Box P$ and $\Box P \models_{\overline{\mathrm{LPD}}} P$, but "$P \rightarrow \Box P$" is **not** always valid in LPD;
- $\Box P \models_{\overline{\mathrm{LPD}}} P! \models_{\overline{\mathrm{LPD}}} P? \models_{\overline{\mathrm{LPD}}} \Diamond P$.

It happens, however, that:

2.18 Theorem. The following propositions are not always true:

- $\Diamond P \models_{\overline{\mathrm{LPD}}} P?$;
- $P? \models_{\overline{\mathrm{LPD}}} P!$;
- $P! \models_{\overline{\mathrm{LPD}}} \Box P$.

2.19 Notation. CL is the open version of classical logic.[4]

The following theorem states that LPD is a conservative extension of CL.

2.20 Theorem. If Φ and P are respectively a collection of modality-free formulas of L and a modality-free formula of L, then the following proposition is valid:

- $\Phi \models_{\overline{\mathrm{LPD}}} P$ if, and only if, $\Phi \models_{\overline{\mathrm{CL}}} P$.

2.21 Lemma. Let Γ be a collection of formulas of L of one of the forms Q or $Q?$, such that Q is modality-free, whereon Γ has at least a formula of the form $Q?$. Let P be a modality-free formula of L, and consider $\overline{\Gamma}$ the collection $\{\, Q \in \Gamma \mid Q$ is modality-free $\}$. The following propositions are equivalent:

[4] Open versions of classical logic are presented in [13, 17, 24], whereas closed versions of classical logic are given in [1, 8, 9].

(1) $\Gamma \mathrel{|\!\!\!\overline{\underline{\;\;\;}}}_{\text{LPD}} P?$;

(2) there is a formula $Q? \in \Gamma$ such that $\overline{\Gamma} \cup \{Q?\} \models_{\overline{\text{LPD}}} P?$;

(3) there is a formula $Q? \in \Gamma$ such that $\overline{\Gamma} \cup \{Q\} \models_{\overline{\text{CL}}} P$.

Proof:.
We demonstrate that (1) implies (2), and give to the reader the task of proving that (2) implies (3) and (3) implies (1).
Suppose that $\Gamma \models_{\overline{\text{LPD}}} P?$.
There exists a finite subset $\Gamma' \cup \{Q_1?, \ldots, Q_n?\}$ of Γ such that $\Gamma' \subseteq \overline{\Gamma}$ and $\Gamma' \cup \{Q_1?, \ldots, Q_n?\} \models_{\overline{\text{LPD}}} P?$. If $n = 0$, then there is nothing more to prove, so consider that $n > 0$.
Suppose that, for each $i \in \{1, \ldots, n\}$, it is not the case that $\Gamma' \cup \{Q_i?\} \models_{\overline{\text{LPD}}} P?$. Then, for each $i \in \{1, \ldots, n\}$, there is an LPD-structure $H_i = \langle \Delta, W_i, W_i' \rangle$ for L such that H_i satisfies $\Gamma' \cup \{Q_i?\}$ and H_i does not satisfy $P?$. If H is an LPD-structure for L defined by $H = \langle \Delta, W_1 \cup \ldots \cup W_n, W_1' \cup \ldots \cup W_n' \rangle$, then H satisfies $\Gamma' \cup \{Q_1?, \ldots, Q_n?\}$ but does not satisfies $P?$, which is absurd.
Therefore there is $i \in \{1, \ldots, n\}$ such that $\Gamma' \cup \{Q_i?\} \models_{\overline{\text{LPD}}} P?$. $\square$

LPD is a monotonic logic designed to perform reasoning in multiple scenarios. Given the possible and plausible scenarios, a LPD-structure H represents them in the collections of worlds W and W', respectively. The plausible formula $P?$ holding in H means that there is a plausible scenario in which P holds. The strictly plausible formula $P!$ holding in H means that P holds in all plausible scenarios. Similarly for $\Box P$ and $\Diamond P$, but now they hold in the possible scenarios. A question remains: from where these scenarios emerge? What distinguish them in possible and plausible scenarios? Our proposal to answering this question is presented in section 4.

3. An Axiomatics for Plausible Deduction

Next an axiomatic calculus for LPD is defined. It will be done according to the *open style*, that is, with no restriction for applications of inference rules for introducing the universal quantifier or the necessity connective, but with restrictions for introducing implication. Open versions of classical logic are presented in [13, 17, 24]. In [2, 3] it is given a general method for introducing implication in *open calculi*. The other method for dealing with introduction inference rules, the *closed style*, presented for example in [1, 8, 9], is not used here, because it is not useful for maintaining one of the guidelines taken into account when modeling LPD, namely that this logic must have "$P/P!$" at least as a derived rule, which does not happen in the correspondent *closed version* of LPD.

For getting easier applications of introduction of implication, it is done a kind of tracing of the use of the rules for introducing the universal quantifier or the necessity connective, simply by the use of entities called *varying objects*, which correspond to variables in logics without modalities. When there are modalities, it is necessary at least one additional varying object for indicating a kind of variation along worlds; here, for the sake of convention, it is used the sign "□". The tracing begins by associating to each application of an inference rule a set of varying objects, eventually empty.

3.1 Definition. A *varying object in* LPD is a variable or the sign "□".

The definition below, together with definition 2.14, specifies when a given varying object is free in a formula inside LPD.

3.2 Definition. A formula P is said □-*closed* if P has one of the forms $\Box Q$, $Q!$, $\neg R$, $R \to S$ or $\forall x\, R$, whereon R and S are □-closed; otherwise it is said that □ is free in P.

3.3 Definition. A formula is said to be *modal* if it has some occurrence of the signs "□" or "!"; otherwise it is said to be *modality-free*.

3.4 Definition. $P(x|t)$ is the formula obtained from P by substituting t for each free occurrence of x, and replacing consistently bound occurrences of variables in P for other ones don't occurring in P when it is necessary.[5]

3.5 Definition. The calculus for LPD has the following postulates (axiom schemes and inference rules), whereon, for each inference rule, a varying object can be attached:

(1) $P \to (Q \to P)$;

(2) $(P \to Q) \to \Big((P \to (Q \to R)) \to (P \to R)\Big)$;

(3) $\dfrac{P,\ P \to Q}{Q}$, whereon no varying object is attached;

(4) $(\neg P \to Q) \to ((\neg P \to \neg Q) \to P))$;

(5) $\forall x\, P \to P(x|t)$;

(6) $\forall x\, (P \to Q) \to (\forall x\, P \to \forall x\, Q)$;

(7) $P \to \forall x\, P$, whereon x is not free in P;

(8) $\dfrac{P}{\forall x\, P}$, whereon x is the attached varying object;

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[5] It avoids that occurrences of variables in t become bound in $P(x|t)$. A detailed definition of $P(x|t)$, taking into account that all variables occurring in t must remain free in $P(x|t)$, can be found in [1, 8].

(9) $\Box P \to P$;
(10) $\Box(P \to Q) \to (\Box P \to \Box Q)$;
(11) $P \to \Box P$, whereon $\Box$ is not free in P;
(12) $\dfrac{P}{\Box P}$, whereon $\Box$ is the attached varying object;
(13) $\Box P \to P!$;
(14) $P! \leftrightarrow P$, whereon $\Box$ is not free in P;
(15) $(P! \to P)!$;
(16) $(P \to Q)! \to (P! \to Q!)$;
(17) $\forall x\,(P!) \to (\forall x\, P)!$.

3.6 Definition. As it is usual, a syntactical consequence relation "$\vdash_{\mathrm{LPD}}$" is defined, relating collections of formulas in LPD to formulas in LPD. Beyond that, it is defined "$\vdash^{\mathcal{V}}_{\mathrm{LPD}}$", whereon $\mathcal{V}$ is a collection of varying objects:

- A deduction $\mathcal{D}$ in LPD *depends on* a collection $\mathcal{V}$ (of varying objects) if $\mathcal{V}$ contains the collection of varying objects o of all applications of rules in $\mathcal{D}$ having a hypothesis in which o is free such that there is a formula, justified as a premise in $\mathcal{D}$, whereon o is free too, relevant to this hypothesis in $\mathcal{D}$.
- P is a *syntactical consequence of* Γ *in* LPD *depending on* $\mathcal{V}$ if there is a deduction of P from Γ in LPD depending on $\mathcal{V}$; it is noted by $\Gamma \vdash^{\mathcal{V}}_{\mathrm{LPD}} P$.

3.7 Definition. A formula P is said to be a *thesis of* LPD if $\vdash_{\mathrm{LPD}} P$.

3.8 Definition. A collection Φ of formulas of L is said LPD-*trivial* if, for each formula P of L, $\Phi \vdash_{\mathrm{LPD}} P$.

3.9 Theorem. All signs "$\to$", "$\wedge$", "$\vee$", "$\leftrightarrow$", "$\neg$", "$\forall$" and "$\exists$" behave in LPD like in open classical logic[6]. Below it is formulated a version of the deduction theorem for LPD:

- If $\begin{cases} \Gamma \cup \{P\} \vdash^{\mathcal{V}}_{\mathrm{LPD}} Q, \\ \text{no varying object of } \mathcal{V} \text{ is free in } P, \end{cases}$ then $\Gamma \vdash^{\mathcal{V}}_{\mathrm{LPD}} P \to Q$.

3.10 Theorem. The alethic modalities "$\Box$" and "$\Diamond$" behave in LPD like they do in the open version of S5 logic, that is:

- $\Box P \vdash^{\emptyset}_{\mathrm{LPD}} P$;
- $P \vdash^{\Box}_{\mathrm{LPD}} \Box P$;
- $P \vdash^{\emptyset}_{\mathrm{LPD}} \Diamond P$;
- if Q is $\Box$-closed, then $\Diamond P, P \to Q \vdash^{\Box}_{\mathrm{LPD}} Q$.

[6] As it is presented, for example, in [13, 17, 24]; in [2, 3] general concepts about open calculi, varying objects and deduction theorems are analyzed, together with the consequence relation "$\vdash^{\mathcal{V}}$" and other similar one.

3.11 Theorem. The non alethic modalities "$!$" and "$?$" have respectively the following introduction and elimination rules:

- $P \mathrel{\vert\!\frac{\;\Box\;}{\text{LPD}}} P!\,;$
- if Q is $\Box$-closed, then $P?, P \to Q \mathrel{\vert\!\frac{\;\Box\;}{\text{LPD}}} Q$.

3.12 Theorem. If LPD$'$ is an axiomatic calculus obtained from the axiomatic calculus for LPD by adding the axiom scheme "$P! \to P$", then the following propositions are true:

- $\vert\!\frac{\quad}{\text{LPD}'} P \to P?;$
- $\vert\!\frac{\quad}{\text{LPD}'} P! \leftrightarrow \Box P;$
- $\vert\!\frac{\quad}{\text{LPD}'} P? \leftrightarrow \Diamond P;$
- $\Gamma \mathrel{\vert\!\frac{\;\mathcal{V}\;}{\text{LPD}'}} P$ if, and only if, $\Gamma! \mathrel{\vert\!\frac{\;\mathcal{V}\;}{\text{LPD}}} P!$, whereon $\Gamma! = \{\, P! \mid P \in \Gamma \,\}$.

3.13 Theorem. The following propositions show the interrelationship among necessity, skeptical plausibility, credulous plausibility and possibility in LPD:

- $\vert\!\frac{\quad}{\text{LPD}} \Box P \to P!\,;$
- "$P!/\Box P$" is not a valid rule in LPD;
- $\vert\!\frac{\quad}{\text{LPD}} P! \to P?\,;$
- "$P?/P!$" is not a valid rule in LPD;
- $\vert\!\frac{\quad}{\text{LPD}} P? \to \Diamond P\,;$
- "$\Diamond P/P?$" is not a valid rule in LPD.

The semantical and syntactical consequence relations of LPD are equivalent.

3.14 Theorem (correctness and completeness of LPD)**.**

- $\Gamma \mathrel{\vert\!\frac{\quad}{\text{LPD}}} P$ if, and only if, $\Gamma \mathrel{\vert\!\overline{\frac{\quad}{\text{LPD}}}} P$.

Proof:.
An auxiliary three-sorted quantificational logic, LPD$'$, with no modalities, with a semantics and an axiomatic calculus, is constructed.
A language L' of LPD$'$ is constructed from a language L of LPD according to the following conditions:

(1) each constant of L is still a constant of L';
(2) each n-ary function sign of L is still an n-ary function sign of L';
(3) all variables of L are still variables of L', called *normal variables of* L', but there is one more *extra variable* v, ranging over all possible worlds;
(4) for each predicate sign p of L, p is an $n+1$-ary predicate sign of L';
(5) there are only two primitive connectives in L': "$\to$" and "$\neg$";
(6) there are three primitive quantifiers in L': "$\forall$", "$\forall_\Box$" and "$\forall_!$".

The terms and formulas in LPD$'$ are defined according to the following clauses:

(1) each term of L is a term of L';

(2) the additional variable v of L' is a term of L';
(3) if p is an n-ary predicate sign of L and $t_1, \dots, t_n$ are terms of L, then p is an $n+1$-ary predicate sign of L', $t_1, \dots, t_n, v$ are terms of L' and $p(t_1, \dots, t_n, v)$ is a formula of L';
(4) if P, Q are formulas of L', then $\neg P$ and $P \rightarrow Q$ are formulas of L';
(5) if x is a normal variable of L' and P is a formula of L', then $\forall x\, P$ is a formula of L';
(6) if P is a formula of L', then $\forall_{\Box} v\, P$ and $\forall_{!} v\, P$ are formulas of L'.

A world over Δ for L' is just a world over Δ for L. An LPD$'$-structure for L' is just an LPD-structure for L. An LPD$'$-interpretation Θ for L' is a quadruple $\langle \Delta, W, W', s \rangle$, whereon $\langle \Delta, W, W' \rangle$ is an LPD$'$-structure for L', and s is a function defined for all variables of L', such that s associates each normal variable to an element of Δ, and associates the extra variable v to an element of W. Given an LPD$'$-interpretation $\Theta = \langle \Delta, W, W', s \rangle$ for L', the functions Θ_{D} and Θ_{E}, called respectively the *denotation for L' defined by* Θ and *the evaluation for L' defined by* Θ, are specified in an analogous way as it was done in definition 2.8, with the following differences:
(1) if p is an $n+1$-ary predicate sign in L', and $t_1, \dots, t_n$ are terms in L (so also in L'), then
$\Theta_{\mathrm{E}}(p(t_1, \dots, t_n, v)) = 1$ iff $\langle \Theta_{\mathrm{D}}(t_1), \dots, \Theta_{\mathrm{D}}(t_n) \rangle \in s(v)(p)$;
(2) $\Theta_{\mathrm{E}}(\forall_{\Box} v\, P) = \min\{\Theta(v|w)_{\mathrm{E}}(P) \mid w \in W\}$;
(3) $\Theta_{\mathrm{E}}(\forall_{!} v\, P) = \min\{\Theta(v|w)_{\mathrm{E}}(P) \mid w \in W'\}$.

An LPD$'$-valuation for L' defined by an LPD$'$-structure H for L' is specified in a same way as it was done in definition 2.9, and finally a semantics for LPD$'$ is specified in an identical way as it was done in definition 2.10.

LPD$'$ is a three-sorted logic, in which, for a given LPD$'$-structure $H = \langle \Delta, W, W' \rangle$ for L', the sorts are Δ, W and W', whereon a normal variable ranges over Δ, whereas the only extra variable v ranges over the collection W of possible worlds. The universal quantifier "$\forall_{!} v$" obliges v to range only over W', the collection of plausible worlds of H.

For each pair of corresponding languages L and L' for the logics LPD and LPD$'$, let f be a function from L to L', defined through the following clauses:
(1) if p is an n-ary predicate sign in L, and $t_1, \dots, t_n$ are terms in L, then $f(p(t_1, \dots, t_n)) = p(t_1, \dots, t_n, v)$;
(2) $f(\neg P) = \neg f(P)$;
(3) $f(P \rightarrow Q) = f(P) \rightarrow f(Q)$;
(4) $f(\forall x\, P) = \forall x\, f(P)$;
(5) $f(\Box P) = \forall_{\Box} v\, f(P)$;
(6) $f(P!) = \forall_{!} v\, f(P)$.

A calculus for LPD$'$ is defined just writing all postulates of the calculus for LPD in a language for LPD$'$, taking into account the translation f.

It is not difficult to prove that, given a collection Γ of formulas of L and a formula P of L, if $\Gamma' = \{f(Q) \mid Q \in \Gamma\}$ and $P' = f(P)$, then the following propositions are valid:
- $\Gamma \mid\!\frac{}{\text{LPD}} P$ if, and only if, $\Gamma' \mid\!\frac{}{\text{LPD}'} P'$; (1)
- $\Gamma \models_{\text{LPD}} P$ if, and only if, $\Gamma' \models_{\text{LPD}'} P'$. (2)

It is easy to prove that the calculus for LPD$'$ is correct and complete with respect to the semantics for LPD$'$, that is:
- $\Gamma \mid\!\frac{}{\text{LPD}'} P$ if, and only if, $\Gamma \models_{\text{LPD}'} P$. (3)

From propositions (1), (2) and (3), it follows finally that the calculus for LPD is correct and complete with respect to LPD semantics:
- $\Gamma \mid\!\frac{}{\text{LPD}} P$ if, and only if, $\Gamma \models_{\text{LPD}} P$.

□

The following propositions state respectively syntactical versions of theorem 2.20 and lemma 2.21.

3.15 Theorem. If Φ and P are respectively a collection of modality-free formulas of L and a modality-free formula of L, then the following proposition is valid:
- $\Phi \mid\!\frac{}{\text{LPD}} P$ if, and only if, $\Phi \mid\!\frac{}{\text{CL}} P$.

Proof:. It follows directly from theorems 2.20 and 3.14. □

3.16 Lemma. Let Γ be a collection of formulas of L of one of the forms Q or $Q?$, such that Q is modality-free, whereon Γ has at least a formula of the form $Q?$. Let P be a modality-free formula of L, and consider $\overline{\Gamma}$ the collection $\{Q \in \Gamma \mid Q \text{ is modality-free}\}$. The following propositions are equivalent:

(1) $\Gamma \mid\!\frac{}{\text{LPD}} P?$;

(2) there is a formula $Q? \in \Gamma$ such that $\overline{\Gamma} \cup \{Q?\} \mid\!\frac{}{\text{LPD}} P?$;

(3) there is a formula $Q? \in \Gamma$ such that $\overline{\Gamma} \cup \{Q\} \mid\!\frac{}{\text{CL}} P$.

Proof:. It follows directly from lemma 2.21 and theorem 3.14. □

4. The Logic of Plausible Reasoning

LPD is a logic to deal with deductive reasoning in a multiple scenarios environment, the so called *plausible scenarios.* The key question to be discussed now is how to construct these plausible scenarios. How do they come into being in the scope of assumptions and facts to be reasoned about? They come, naturally, from our imagination, from our capacity (maybe a professionally trained skill) of producing good guesses by hypothesizing, making conjectures, and devising multiple alternative states of affairs which deserve examination, as does one playing chess or analyzing possible investments in an economic environment, or elaborating scientific explications for experimental phenomena. The role we believe a logic ought to play in such circumstances is to provide the means to represent those conjectures and to enable their inferential analysis.

In our formalization of reasoning we distinguish *hard* from *soft* premises. Hard premises are taken for granted, admitted as true, at least for the sake of providing a common ground for reasoning and discourse. As Popper [22] more than once remarked it, it is not possible to doubt of everything at the same time. These hard premises are made out of facts and principles we are not inclined to doubt, and are therefore taken as assured knowledge. Soft premises are how hypotheses, conjectures and guesses, the putative assertions that are in test and that generate alternative scenarios, are represented. Alternative scenarios are so produced because conjectures may, and often do, conflict with each other. In the most flagrant case one can be the direct denial of the other. In spite of it, we may wish to regard both as plausible for the sake of analysis and further inquire. An important feature of reasoning, which is contemplated in the formalization proposed here, is precisely the accommodation of the conflicting conjectures composing alternative scenarios into a same logical framework without carrying out a logical catastrophe.

Hard premises are sufficient to determine possible worlds and scenarios, for a possible world is simply any model of them, whereas a possible scenario is any of its consistent superset of propositions. Hence, possible assertions are the ones consistent with the hard premises. But how plausible worlds and scenarios are obtained? As we said, out of conjectures. Some conjectures may be mutually compatible, meaning that they can be simultaneously held without contradiction, i.e., they may coexist in a same scenario. A (maximal) collection of compatible conjectures gives rise to a plausible scenario; a plausible world is a world that in addition to satisfy the hard premises also satisfies a plausible scenario. The conflicting conjectures, on the other hand, are disposed in different scenarios, reflecting the complexities of real reasoning. The logic we propose is able to accommodate those scenarios in a same unity, and so to analyze certain features about propositions that make sense just against this framework. For instance, we may say that a proposition is *weakly plausible*, or simply *plausible* if it is held in at least one plausible scenario, while we call a proposition *strictly plausible* if it is held in all plausible scenarios. An assertion is plausible not only because it is consistent with the hard premises but because, in addiction to this, there is a bulk of conjectures (positive reasons) supporting it.

The presentation of the technical details of the ideas just described, and how they are mathematically formulated and developed, follows.

The premises, forming what we could call an LPR-basis (LPR stands for *Logic of Plausible Reasoning*), are presented in two sets. The first is a set of first order formulas notating the *hard premises* and the second is a set of *soft premises*, formulas in an extended notation, that we call *generalizations*, which represent the conjectures. Generalizations are, in fact, a fairly flexible way to represent conjectures, since they allow a conjecture so presented to bear conditions limiting its scope of application. In other words, they admit the annotation of exceptions to, not so, general conjectures.

The premises of an LPR-basis are scrutinized in order of detecting and grouping compatible generalizations. This is done with the help of the notion of *extension* to be constructed from the data of the basis, given in definition 4.18. (Later, in section 5, we present alternative ways to reach an equivalent construction. We hope that this plurality of styles may contribute to the clarification of this work.) Compatible conjectures — extracted from the generalizations and always signalized with a ? mark which indicates they are plausible formulas — are conjoined and added to the extension. A *plausible scenario* (definition 4.32) is a deductively closed set of modality-free formulas generated by the hard premises together with the conjectures extracted from a maximal collection of compatible generalizations. Strictly plausible sentences ($P!$) are the ones derived from generalizations, which belong to all plausible scenarios (the definition 4.31 of strongly triggered generalization captures this feature). *Plausible worlds* are, thus, the classical models of the plausible scenarios. Therefore, plausible sentences are held in some plausible scenarios and thus true in some plausible worlds, whereas strictly plausible sentences are held in all plausible scenarios and thus true in all plausible worlds. We insist that models for the hard premises form the possible worlds, and hence any plausible world is also a possible world. Possible sentences ($\Diamond P$) are the ones consistent with the hard premises, they hold in some possible scenarios (and worlds). Necessary sentences ($\Box P$) are logical consequences of the hard premises, they hold in all possible scenarios (and worlds). The set of theorems (provable sentences) of an LPR-theory is the deductive closure in LPD of the theory formed by the hard premises, the possible sentences, the plausible and the strictly plausible sentences.

Therefore, when all this logical treatment is performed with the initial LPR-basis, we end up with a theory describing multiple scenarios which can be treated, both semantic and syntactically, in the logic LPD for plausible deduction presented in sections 2 and 3. The theory in LPD formed from an LPR-basis is given by the definition 4.39, and the LPD-structure satisfying it by the definition 4.43.

Technicalities are in order. They follow.

4.1 Definition. A *generalization* (*in* L) is an expression of the form "$P \multimap Q$" such that P, Q are modality-free formulas (of L)[7]; in such expression P is called *the conjecture* and Q *the restriction* or *exception of* this generalization. An *instance of a generalization* $P \multimap Q$ (*in* L) is an expression $P' \multimap Q'$, whereon P', Q' are *consistent instances of* P, Q (*in* L)[8].

4.2 Reading. The intended meaning of a generalization "$P \multimap Q$" is that P represents a conjecture that holds under the condition that the restriction Q does not. It is read as "generally P, unless Q".

4.3 Definition. $P \multimap \;\rightleftharpoons P \multimap \bot$.

4.4 Definition. An LPR-*basis* (in L) is a pair $\tau = \langle T, G\rangle$, whereon T is a collection of modality-free formulas (of L) and G is a collection of generalizations (in L).[9] T and G represent respectively the *hard* and *soft premisses* of the LPR-basis τ.

4.5 Definition. A collection of instances of generalizations of G in L is said to be a *rule in* τ. A finite collection of instances of generalizations of G in L is said to be a *finite rule in* τ.

4.6 Notation. Unless stated otherwise, from now on, in the remaining of this paper:
- $\tau = \langle T, G\rangle$ is an LPR-basis in L;
- the letter G followed by primes and/or subscripts denotes a rule in τ.

4.7 Definition. Given a rule G' in τ, it is specified:
- $\mathrm{Conj}(G')$[10] $\rightleftharpoons \{P \mid$ there exists Q such that "$P \multimap Q$" belongs to $G'\}$;
- $\mathrm{Rest}(G')$[11] $\rightleftharpoons \{Q \mid$ there exists P such that "$P \multimap Q$" belongs to $G'\}$.

4.8 Definition. If P is a formula and $x_1, \ldots, x_n$ are the variables free in P, then:
- $\mathrm{uc}(P)$, *the universal closure of* P, is the formula $\forall x_1 \ldots \forall x_n\, P$;
- $\mathrm{ec}(P)$, *the existential closure of* P, is the formula $\exists x_1 \ldots \exists x_n\, P$.

4.9 Definition. If $\Phi = \{P_1, \ldots, P_n\}$ is a finite collection of formulas of L, then:
- $\bigwedge \Phi \rightleftharpoons \mathrm{uc}(P_1 \wedge \ldots \wedge P_n)$;
- $\bigvee \Phi \rightleftharpoons \mathrm{ec}(P_1 \vee \ldots \vee P_n)$.

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[7] The notation established in 2.2, p. 9, continues to hold. A modality-free formula is defined in 3.3, p. 14. It means that a formula does not contain the signs "$\Box$" or "!".

[8] That is, variables occurring both in P and Q are replaced by the same terms (in L).

[9] L is a fixed language for LPD whose alphabet has all constants, function and predicate signs occurring both in T and in G, and in all possible conclusions one wants to extract from an LPR-basis $\langle T, G\rangle$.

[10] "$\mathrm{Conj}(G')$" is read "*conjectures of* G'".

[11] "$\mathrm{Rest}(G')$" is read "*restrictions of* G'".

4.10 Scholium. The following propositions are valid:
- $\bigwedge \emptyset = \top$;
- $\bigvee \emptyset = \bot$.

4.11 Lemma. Let Φ be a finite collection of formulas. The following propositions are valid:
- $\Gamma \cup \bigwedge \Phi \mid\!\frac{\ }{\text{LPD}} P$ if, and only if, $\Gamma \cup \Phi \mid\!\frac{\ }{\text{LPD}} P$;
- $\Gamma \cup \bigwedge \Phi \mid\!\frac{\ }{\text{CL}} P$ if, and only if, $\Gamma \cup \Phi \mid\!\frac{\ }{\text{CL}} P$.[12]

4.12 Reading. Inside an LPR-basis $\tau = \langle T, G \rangle$, there is a holistic way for reading generalizations, since the behavior of each one of them depends on itself and on all other ones. For each finite rule G' in τ, G' can be read "$\bigwedge \mathrm{Conj}(G')$ is a conjecture in τ, unless $\bigvee \mathrm{Rest}(G')$ is the case".

Information which comes from rules is marked by an "?". Since a rule might be an infinite collection of instances of generalizations, the information carried in its conjectures, and also in the restrictions it is subject to, is extracted using finite subrules. We call them, respectively, *hypotheses* and *limits* of the rule.

4.13 Definition.
- $\mathrm{Hyp}(G')$[13] $\rightleftharpoons \{(\bigwedge \mathrm{Conj}(G''))? \mid G''$ is finite and $G'' \subseteq G'\}$;
- $\mathrm{Lim}(G')$[14] $\rightleftharpoons \{(\bigvee \mathrm{Rest}(G''))? \mid G''$ is finite and $G'' \subseteq G'\}$.

4.14 Scholium. The following propositions are valid:
- $\mathrm{Hyp}(\emptyset) = \{\top?\}$;
- $\mathrm{Lim}(\emptyset) = \{\bot?\}$.

Next, it is defined the key concept of an *extension in an* LPR-basis τ. Extensions are LPD-deductively closed sets of formulas that complement the assured knowledge with conjectures extracted from the generalizations. The generalizations are scrutinized as a whole in order to check out whether to include them into the extension. The inclusion of one generalization depends on the complete theory (derived from hard premises and generalizations) since it must be verified that its restriction cannot be derived in the extension. Moreover, they are not taken one by one but in sets of instances of generalizations, or rules as defined in 4.5, with two integrity constraint conditions: their hypotheses must be consistent with the hard premises in T, and their limits must not be derived in the extension. In more technical terms, we say that,

[12] For the second proposition, we are also considering that Γ and Φ are collections of modality-free formulas, and that P is a modality-free formula.
[13] "$\mathrm{Hyp}(G')$" is read "*hypotheses of* G'".
[14] "$\mathrm{Lim}(G')$" is read "*limits of* G'".

for each rule G' in τ, $\mathrm{Hyp}(G')$ is included into an extension if none of its limits is proved in it. The fixed point construction in definition 4.18 reflects the non local character of extensions[15], therefore it is not a constructive definition.

4.15 Definition.
- $\mathrm{Th}_{\mathrm{LPD}}(\Phi) \rightleftharpoons \{P \mid P$ is a formula of L and $\Phi \vdash_{\mathrm{LPD}} P\}$;
- $\mathrm{Th}_{\mathrm{CL}}(\Phi) \rightleftharpoons \{P \mid P$ is a modality-free formula of L and $\Phi \vdash_{\mathrm{CL}} P\}$, whereon Φ is a collection of modality-free formulas of L.

4.16 Definition. $\Psi_\tau(\Phi)$ and $\overline{\Psi}_\tau(\Phi)$ are respectively the least collections of formulas of L and of rules in τ satisfying the following conditions:
(1) $T \subseteq \Psi_\tau(\Phi)$;
(2) if $\Psi_\tau(\Phi) \vdash_{\mathrm{LPD}} P$, then $P \in \Psi_\tau(\Phi)$;
(3) if $\mathrm{Lim}(G') \cap \mathrm{Th}_{\mathrm{LPD}}(\Phi \cup \mathrm{Hyp}(G')) = \emptyset$, then $\mathrm{Hyp}(G') \subseteq \Psi_\tau(\Phi)$ and $G' \in \overline{\Psi}_\tau(\Phi)$.

Notice that condition (3) checks whether the limits of a rule are not proved in LPD from Φ and its own hypotheses.

4.17 Scholium. $\overline{\Psi}_\tau(\Phi) = \{G' \mid \mathrm{Lim}(G') \cap \mathrm{Th}_{\mathrm{LPD}}(\Phi \cup \mathrm{Hyp}(G')) = \emptyset\}$.

4.18 Definition. A collection E of formulas of L is said an *extension in* τ if $\Psi_\tau(E) = E$; in this case the collection $\overline{\Psi}_\tau(E)$ is denominated the *set of generating rules of* E *in* τ.

4.19 Example. Let

$$T = \emptyset,$$
$$G = \{\, \mathrm{flies}(x) \prec \mathrm{penguin}(x)\, ,\, \mathrm{feathered}(x) \prec \mathrm{chick}(x) \,\}.$$

T and G form an elementary LPR-basis telling about birds. As it cannot be proved from τ that there is at least a penguin or that there is at least a chick, we have that

$$E = \mathrm{Th}_{\mathrm{LPD}}\Big(\Big\{ \Big(\forall x\, (\mathrm{flies}(x) \wedge \mathrm{feathered}(x))\Big)? \Big\}\Big)$$

is the only extension in τ.

4.20 Example. Considerer another LPR-basis that is almost equal to the one just given above, but with

$$T = \{\, \mathrm{penguin}(\mathrm{Tweety}), \mathrm{chick}(\mathrm{Woody}) \,\},$$

[15] The term "extension" and the definition through fixed points follow the general line of the original paper of Reiter on default logic [23]. In section 5, we present some equivalent notions playing the same role as extensions which do not appeal to fixed point constructions.

whereon G remains as in the example above. Now, as it can be proved from T that Tweety is a penguin, and that Woody is a chick, the instances

$$\text{flies(Tweety)} \multimap \text{penguin(Tweety)}$$
and
$$\text{feathered(Woody)} \multimap \text{chick(Woody)}$$

are blocked, whereas the instances

$$\text{flies(Woody)} \multimap \text{penguin(Woody)}$$
and
$$\text{feathered(Tweety)} \multimap \text{chick(Tweety)}$$

can still be applied, so the only extension in τ equals

$$\mathrm{Th}_{\mathrm{LPD}}\Big(T \cup \{\ (\text{flies(Woody)} \wedge \text{feathered(Tweety)})?\ \}\Big).$$

4.21 Example. An extension in τ might change according to the language being considered. For instance, consider an LPR-basis whereon

$$T = \{\text{penguin(Tweety)}\},$$
$$G = \{\text{flies}(x) \multimap \text{penguin}(x)\}.$$

If L is the language formed with the non-logical symbols which appear in the basis (as it has been implicitly assumed in the examples above) then $E = \mathrm{Th}_{\mathrm{LPD}}(T)$ is the only extension. However, let the language of τ includes a unary functional symbol "f" and infinitely many constant symbols ("Tweety" and, for each $i \geq 1$, "c_i"), besides the predicate symbols "flies" and "penguin". Now, the extension in τ is given by

$$E = \mathrm{Th}_{\mathrm{LPD}}\left(T \cup \left\{\left(\bigwedge\{\text{flies}(t_1), \ldots, \text{flies}(t_n)\}\right)? \;\middle|\; n \geq 1\right\}\right),$$

whereon each t_i is any term distinct from "Tweety" and from each variable. The terms t_i cannot be a variable because the formulas added to E are universally closed, and this will mean that every individual flies, which is not true since we are not allowed to infer that Tweety flies (as far as the theory of the example goes, we are not allowed to infer that Tweety does not fly either). Both assertions "Tweety flies" and "Tweety does not fly" are possible but not plausible assertions. In the logic LPR, as we will see in definition 4.39, "$\Diamond$ flies(Tweety)" and "$\Diamond\neg$ flies(Tweety)" are theorems of the basis τ, however neither "flies(Tweety)?" nor "$\neg$ flies(Tweety)?" are.

The following theorems state some trivial results about limit cases.

4.22 Theorem. The following propositions are equivalent:
- $\overline{\Psi}_\tau(\Phi) \neq \emptyset$;
- $\emptyset \in \overline{\Psi}_\tau(\Phi)$;
- Φ is not LPD-trivial.

4.23 Theorem.
- If $\overline{\Psi}_\tau(\Phi) = \emptyset$ or $\overline{\Psi}\tau(\Phi) = \{\emptyset\}$, then $\Psi_\tau(\Phi) = \mathrm{Th}_{\mathrm{LPD}}(T)$.

4.24 Theorem. The following propositions are equivalent:
- $\Psi_\tau(\Phi) = \mathrm{Th}_{\mathrm{LPD}}(T)$;
- for each $G' \in \overline{\Psi}_\tau(\Phi)$, $\mathrm{Hyp}(G') \subseteq \mathrm{Th}_{\mathrm{LPD}}(T)$;
- $\mathrm{Hyp}(\overline{\Psi}_\tau(\Phi)) \subseteq \mathrm{Th}_{\mathrm{LPD}}(T)$.

4.25 Definition. It is said that G_1 *is subsumed by* G_2 if there is G_0 such that $G_0 \subseteq G_2$ and G_1 is obtained from G_0 by instantiating simultaneously, in a consistent way[16], free variables by terms in L. When G_1 is subsumed by G_2, it is noted by $G_1 \preceq G_2$, and it is said that G_1 is a *subrule of* G_2. G_1 is said to be *equivalent to* G_2, and it is noted by $G' \approx G''$, if $G_1 \preceq G_2$ and $G_2 \preceq G_1$.

4.26 Definition. Let E be an extension in τ. G' is said *to be inside* E *in* τ if $G' \in \overline{\Psi}_\tau(E)$. If there is an extension E in τ such that G' is inside E in τ, it is said that G' is *compatible in* τ, or that G' is a *collection of (mutually) compatible generalizations in* τ. G' and G'' are said to be *co-extensional in* τ if they are inside a same extension in τ. If G' and G'' are co-extensional in τ but $G' \cup G''$ is not inside any extension in τ, then we say that G' *conflicts with* G'' *in* τ, or that G' and G'' are *conflicting in* τ, or still that some generalizations inside G' *conflict with* some generalizations inside G'' *in* τ.

4.27 Scholium.
- If E is an extension in τ, then $\overline{\Psi}_\tau(E) = \{\, G' \mid G' \text{ is inside } E \text{ in } \tau \,\}$.

4.28 Lemma. The following propositions are valid:
- if $G' \in \overline{\Psi}_\tau(\Phi)$ and $G'' \preceq G'$, then $G'' \in \overline{\Psi}_\tau(\Phi)$;
- if G' is inside an extension E in τ, then each subrule of G' is also inside E in τ.

4.29 Definition. Let E be an extension in τ such that G' is inside E in τ. G' is *maximal inside* E *in* τ if, for all G'' inside E in τ such that $G' \preceq G''$, $G' \approx G''$.

[16] That is, occurrences of the same variable, even occurring in distinct generalizations, must be replaced by the same terms.

4.1. Well Behaved Theories

LPR was designed to produce only one extension for normal, well written bases. The reason for that stems from the fact that LPR can accommodate opposite conjectures into a same extension using the operator "?" for plausibility. Naturally, the opposite conjectures are part of different plausible scenarios. If there is none or more than one LPR-extension, it is because the theory is *defective* in the sense that either a generalization is involved in the derivation of its own restriction or two generalizations are involved in each other restriction. Any of these two situations characterize what we call a *cycle*, a *self-defeating* and a *mutual cycle*, respectively. There is something subtle about the use of information subject to exceptions. Exceptions convey a sort of "meta-knowledge" about the usage of the corresponding information. As such, it (the corresponding information) cannot interfere with the derivation of its own exception. In our formalism, generalizations (in fact, rules, as defined in 4.5) represent "information subject to exceptions", where the restriction of a generalization (or the limits of a rule) represent the exceptions to the information carried out by its conjectures (hypotheses in case of rules). Therefore, in our approach, the limits of the rules induce a hierarchy among them. The limits of a rule must be derived independently of the corresponding hypotheses. There is an analogy with set theory here: the elements of a set must be given before a set is constructed. This is the reason why in Zermelo-Fraenkel set theory the membership relation is irreflexive and antisymmetric. This frequently neglected analysis about how to treat exceptions in ampliative reasoning plays a central role in our approach. It is important not only to characterize defective theories through the detection of cycles, but, mainly, to how extensions are determined in LPR. Following nomenclature adopted in [18], extensions in LPR are said to comply with the *exceptions-first criterion*. In the frequent event that the hypotheses of a rule lead to the derivation of a limit of another rule, only a plausible scenario emerges of the interplay of these rules: the one generated by the first rule (the second rule being precluded by the derivation of its restriction). The alternative scenario where the application of the second rule would block the derivation of its restriction violates the above consideration and so it is prevented in LPR. This is an important feature distinguishing LPR from the seminal approaches to nonmonotonic reasoning in Artificial Intelligence, namely Circumscription [16] and Default Logic [23]. As a matter of fact, to this date, the authors do not know any formalism to nonmonotonic reasoning which gives to the hierarchy among rules induced by their exceptions the recognition and the importance it deserves.

In LPR, mutual cycles might give rise to multiple extensions and self defeating cycles might cause a theory to have no extension. A well behaved theory, one with no cycles, has always only one extension. We investigate on defective theories in [15], whereon some results concerning well behaved theories are presented. The theory in example 4.35 has a mutual cycle, while the theories in examples 4.36, 4.37 and 4.38 present self-defeating cycles.

4.2. Plausible Scenarios

At this point we would like to say that *plausible scenarios* consist of the formulas in T jointed with the conjectures taken from maximal collections of compatible generalizations. This is indeed the case for theories with only one extension, the well behaved theories. In this case, each generalization in the generating set of an extension (jointly with the hard premises in T) gives rise to a plausible scenario. For the sake of generality and uniformity of treatment, though we consider theories with defective cycles and the generalizations which cause them ill conceived, our definition of plausible scenarios works in the general case where theories have more than one extension. Plausible scenarios are, then, constructed taking into account only compatible collections of generalizations appearing in all extensions, the *triggered rules* as defined below.

4.30 Definition. G' is said *triggered in* τ if G' is inside E in τ, for each extension E in τ. G' is said *maximal triggered in* τ if G' is triggered in τ and, for each G'' triggered in τ, if $G' \preceq G''$, then $G' \approx G''$.

4.31 Definition. A generalization is said *strongly triggered in* τ if it belongs to each G' maximal triggered in τ.

Plausible scenarios are now defined in the general case, regardless how well the theory is constructed.

4.32 Definition. A *plausible scenario* Σ *in* τ is a set of formulas such that $\Sigma = \mathrm{Th}_{\mathrm{CL}}(T \cup \mathrm{Conj}(G'))$, whereon G' is maximal triggered in τ.[17] P *holds in a plausible scenario* Σ *in* τ if $P \in \Sigma$. A *plausible world in* τ is any world satisfying a plausible scenario in τ.

Let us now present some examples to show how plausible scenarios are constructed from an LPR-basis τ.

4.33 Example. Suppose we are willing to consider the following information:
- Swedish in general are not Catholic.
- Pilgrims to Lourdes in general are Catholic.
- Joseph is a Swedish who made a pilgrimage to Lourdes.

Which are the plausible scenarios?

[17] The definition of extension guarantees that a plausible scenario is a consistent set of modality-free formulas.

The LPR-basis $\tau = \langle T, G \rangle$ representing the information is given below:

$$T = \{\ \text{Swedish}(\text{Joseph}), \text{pilgrim}(\text{Joseph})\ \};$$

$$\begin{aligned} G = \{\ & (\text{Swedish}(x) \to \neg\, \text{Catholic}(x)) \prec\!\!\!(\ , \\ & (\text{pilgrim}(x) \to \text{Catholic}(x)) \prec\!\!\!(\ \}. \end{aligned}$$

The only extension for τ is $\text{Th}_{\text{LPD}}(T \cup \Phi)$, whereon

$$\begin{aligned} \Phi = \Big\{\ & \Big(\forall x\ (\text{Swedish}(x) \to \neg\, \text{Catholic}(x))\Big)?, \\ & \Big(\forall x\ (\text{pilgrim}(x) \to \text{Catholic}(x))\Big)?\ \Big\}. \end{aligned}$$

The two generalizations in G are incompatible and there are two plausible scenarios:

$$S_1 = \text{Th}_{\text{CL}}\Big(T \cup \{\ \forall x\ (\text{Swedish}(x) \to \neg\, \text{Catholic}(x))\ \}\Big);$$

$$S_2 = \text{Th}_{\text{CL}}\Big(T \cup \{\ \forall x\ (\text{pilgrim}(x) \to \text{Catholic}(x))\ \}\Big).$$

In the first scenario it is conjectured that Joseph is not Catholic; in the second, Joseph is Catholic. Both assertions are plausible.

4.34 Example. Exceptions-first criterion. Suppose we are willing to consider the following information:

- Birds in general fly, unless they are penguins.
- Penguins in general do not fly.
- Tweety and Woody are birds.
- There is inconclusive evidence that Tweety is a penguin.

Which are the plausible scenarios?

The LPR-basis $\tau = \langle T, G \rangle$ representing the information is given below:

$$T = \{\ \text{bird}(\text{Tweety}), \text{bird}(\text{Woody})\ \};$$

$$\begin{aligned} G = \{\ & (\text{bird}(x) \to \text{flies}(x)) \prec\!\!\!(\ \text{penguin}(x)\ , \\ & (\text{penguin}(x) \to \neg\, \text{flies}(x)) \prec\!\!\!(\ , \\ & \text{penguin}(\text{Tweety}) \prec\!\!\!(\ \}. \end{aligned}$$

The only extension for τ is $\text{Th}_{\text{LPD}}(T \cup \Gamma)$, whereon Γ is

$$\Bigg\{ \Bigg(\forall x\ \Big(\text{penguin}(\text{Tweety}) \wedge (\text{penguin}(x) \to \neg\, \text{flies}(x)) \\ \wedge \big(\text{bird}(\text{Woody}) \to \text{flies}(\text{Woody})\big)\Big)\Bigg)?\ \Bigg\}.$$

The only plausible scenario is $\text{Th}_{\text{CL}}(T \cup \Phi)$, whereon Φ is

$$\Big\{\ \forall x\ \Big(\text{penguin}(\text{Tweety}) \wedge (\text{penguin}(x) \to \neg\, \text{flies}(x)) \\ \wedge \big(\text{bird}(\text{Woody}) \to \text{flies}(\text{Woody})\big)\Big)\ \Big\}.$$

In this sole scenario it is conjectured that Tweety is a penguin and it does not fly, while Woody flies and it is not a penguin.

Notice that a scenario where Tweety is not a penguin and consequently flies is counter-intuitive and it is not a plausible scenario in LPR. Why one would assume that Tweety is not a penguin if there is evidence that he is? The only reason is if one considers that the evidence would come from the interplay of the first two generalizations in the example. In this case the conjecture of the first rule intervenes in the non derivation of its own exception. This is exactly what the exceptions-first criterion [18] precludes, and this explains why this is not an acceptable scenario in LPR. The seminal nonmonotonic logics Circumscription [16] and Default Logic [23] do not comply with the exceptions-first criterion, and in the formalization of this example they would yield this alternative scenario. That is the reason why these logics derive "anomalous extensions" in some representations of the frame problem [10]. At the end of the eighties, this issue was in the core of a lively polemic among Artificial Intelligence scientists on the adequacy of nonmonotonic logics to formalize common sense reasoning. A detailed analysis of this question was done by one of the authors in [18].

Some LPR-bases have more than one single extension.

4.35 Example. Let $T = \emptyset$ and $G = \{ p \prec q \,,\, q \prec p \}$.
This basis has two extensions:

$$E_1 = \mathrm{Th}_{\mathrm{LPD}}(\{ p? \})$$
and
$$E_2 = \mathrm{Th}_{\mathrm{LPD}}(\{ q? \}).$$

This basis is defective in the sense that the two generalizations mutually reject each other, one leading to the restriction of the other and vice-versa; this characterizes a *mutual cycle*. In our view this is meaningless. Accordingly to our approach, neither $\mathrm{Th}_{\mathrm{CL}}(\{ p \})$ nor $\mathrm{Th}_{\mathrm{CL}}(\{ q \})$ form a plausible scenario. Plausible scenarios are made out of conjectures present in all extensions.

Some LPR-bases have no extension.

4.36 Example. If $T = \emptyset$ and $G = \{ p \prec q \,,\, q \prec r \,,\, r \prec p \}$, then τ has no extension. The resulting theory presents a self-defeating cycle and again it is a defective one. The three generalizations form a cycle blocking the use of any of them. These generalizations convey any relevant information, or they simply reveal misconceptions from the proponent of the basis?

Some LPR-bases have an extension whose set of generating rules is reduced to $\{\emptyset\}$.

4.37 Example. If $T = \emptyset$ and $G = \{ p \prec p \}$, then $E = \mathrm{Th}_{\mathrm{LPD}}(\emptyset)$ is the unique extension of τ, and $\mathrm{Th}_{\mathrm{CL}}(\emptyset)$ is its corresponding plausible scenario. Notice that $\overline{\Psi}_\tau(E) = \{\emptyset\}$. Again, this basis is defective, for it presents a generalization leading to its own restriction; this characterizes a *self-defeating cycle*. The generalization "$p \prec p$" is meaningless and of no practical use.

4.38 Example. If $T = \emptyset$ and $G = \{\, p \multimap q,\, p \to q \multimap \,\}$, then there is a single extension E such that $\overline{\Psi}_\tau(E) = \{\, \{\, p \multimap q \,\}, \{\, p \to q \multimap \,\} \,\}$. Again, this basis is defective for it presents a generalization leading to its own restriction.

The *theory generated by an* LPR-*basis* τ is defined next. Notice that, if τ has at least one extension, the formulas in T are necessary, the sentences consistent with T are possible, the plausible formulas hold in some plausible scenario and the strictly plausible formulas hold in all plausible scenarios.

4.39 Definition. *The theory generated by an* LPR-*basis* $\tau = \langle T, G \rangle$, denoted by $\Pi(\tau)$, is the least collection of formulas of L satisfying the following conditions:
- $T \subseteq \Pi(\tau)$;[18]
- If $\Pi(\tau) \mid_{\overline{\text{LPD}}} P$, then $P \in \Pi(\tau)$;[19]
- If P is a modality-free sentence and $T \cup \{P\}$ is not LPD-trivial, then $\Diamond P \in \Pi(\tau)$;[20]
- If G' is finite and triggered in τ, then $(\bigwedge \mathrm{Conj}(G'))? \in \Pi(\tau)$;[21]
- If $P \multimap Q$ is strongly triggered in τ, then $P! \in \Pi(\tau)$.[22]

The elements of $\Pi(\tau)$ are also called *theorems of* τ.

4.40 Theorem. $\Pi(\tau) = \mathrm{Th}_{\mathrm{LPD}}(T \cup T_1 \cup T_2 \cup T_3)$, whereon:
- $T_1 = \{\, \Diamond P \mid P$ is a modality-free sentence and $T \cup \{P\}$ is not LPD-trivial $\}$;
- $T_2 = \{\, (\bigwedge \mathrm{Conj}\,(G'))? \mid G'$ is finite and triggered in $\tau \,\}$;
- $T_3 = \{\, P! \mid$ there exists Q such that $P \multimap Q$ is strongly triggered in $\tau \,\}$.

4.41 Definition. $\tau \mid_{\overline{\text{LPR}}} P \rightleftharpoons P \in \Pi(\tau)$.

4.42 Scholium. The four modalities maintain in LPR a relationship analogous to the one already expressed for LPD in theorem 3.13.

Now we are in a position to define an LPD-structure which allows us to reason with the possible and plausible scenarios induced by the hard and soft premises of a given LPR-theory.

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[18] That is, the hard premises are theorems of τ.

[19] That is, the set of theorems of τ is deductively closed in LPD.

[20] That is, $\Diamond P$ is a theorem of τ, for all modality-free sentences P consistent with T in LPD.

[21] There is a maximal triggered G'' in τ such that $G' \preceq G''$ and $\bigwedge \mathrm{Conj}(G')$ holds in the plausible scenario $\mathrm{Th}_{\mathrm{CL}}(T \cup \mathrm{Conj}(G''))$.

[22] That is, P belongs to all plausible scenarios.

4.43 Definition. An LPD-structure $H = \langle \Delta, W, W' \rangle$ *for* L is said *to satisfy* an LPR-basis τ if the following conditions are fulfilled:

- H satisfies T;[23]
- for each modality-free sentence P, if $T \cup \{P\}$ is non LPD-trivial, then H satisfies $\Diamond P$;[24]
- each plausible scenario in τ is satisfied by some plausible world $w' \in W'$;
- each plausible world $w' \in W'$ satisfies some plausible scenario in τ.

4.44 Definition. A formula P is said to be an LPR-*semantical consequence of* an LPR-basis τ if each LPD-structure H satisfying τ also satisfies P. When it happens, it is noted by $\tau \models_{\overline{\text{LPR}}} P$.

Provability and semantical consequence have the same extension (in set-theoretical terms) for LPR.

4.45 Theorem (Correctness and Completeness of LPR)**.**

- $\tau \vdash_{\text{LPR}} P$ iff $\tau \models_{\overline{\text{LPR}}} P$.

Proof:. Just notice that an LPD-structure satisfies τ if, and only if, it satisfies $T \cup T_1 \cup T_2 \cup T_3$, as defined in theorem 4.40, and that LPD is correct and complete, according to theorem 3.14. $\square$

5. Alternative Notions for Extension

In section 4 the key concept of extension was defined as a fixed point of an operator on sets of formulas. This construction via fixed points was introduced by Reiter in his seminal paper presenting Default Logic [23]. The concept of extension plays a central role in the formalization of complex reasoning presented here since it configures the set of compatible and conflicting conjectures yielding the alternative scenarios. In the sequel, some alternatives for the notion of extension are presented, specially the concept of expansion. We hope that this plurality of styles may contribute to a better understanding of this central concept.

5.1 Definition. A set of rules in τ is said to be a *candidate in* τ.

5.2 Notation. From now on, unless stated otherwise, the letter γ followed or not by primes and/or subscripts denotes a candidate in τ.

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23 That is, the formulas in T are satisfied by all possible worlds $w \in W$.

24 That is, P is satisfied in some possible world $w \in W$.

5.3 Definition.
- $\mathrm{Hyp}(\gamma)$[25] $\rightleftharpoons \{ (\bigwedge \mathrm{Conj}(G''))? \mid G''$ is finite and there is $G' \in \gamma$ such that $G'' \subseteq G' \}$;
- $\mathrm{Lim}(\gamma)$[26] $\rightleftharpoons \{ (\bigvee \mathrm{Rest}(G''))? \mid G''$ is finite and there is $G' \in \gamma$ such that $G'' \subseteq G' \}$.

5.4 Scholium. The following propositions are valid:
- $\mathrm{Hyp}(\gamma) = \bigcup_{G' \in \gamma} \mathrm{Hyp}(G')$;
- $\mathrm{Lim}(\gamma) = \bigcup_{G' \in \gamma} \mathrm{Lim}(G')$.

5.5 Scholium. $\mathrm{Hyp}(\emptyset) = \mathrm{Lim}(\emptyset) = \emptyset$.[27]

5.6 Definition. It is said that G' *is subsumed by* γ, and it is noted by $G' \preceq \gamma$, if, for each finite subset G_1 of G', there is $G_2 \in \gamma$ such that $G_1 \preceq G_2$. It is said that γ *is subsumed by* γ', and it is noted by $\gamma \preceq \gamma'$, if, for each $G' \in \gamma$, $G' \preceq \gamma'$. γ is said to be *equivalent to* γ' *in* τ, and it is noted by $\gamma \approx \gamma'$, if $\gamma \preceq \gamma'$ and $\gamma' \preceq \gamma$.

5.7 Definition. $\Upsilon_\tau(\gamma) \rightleftharpoons \mathrm{Th}_{\mathrm{LPD}}(T \cup \mathrm{Hyp}(\gamma))$.
$\Upsilon_\tau(\gamma)$ is denominated the *set of theorems of* γ *in* τ.

5.8 Lemma. $\Upsilon_\tau(\gamma \cup \{G'\}) = \mathrm{Th}_{\mathrm{LPD}}(\Upsilon_\tau(\gamma) \cup \mathrm{Hyp}(G'))$.

5.9 Definition. A candidate γ in τ is said to be *an expansion in* τ if the following condition is satisfied:
- for all G', $G' \preceq \gamma$ iff $\mathrm{Lim}(G') \cap \Upsilon_\tau(\gamma \cup \{G'\}) = \emptyset$.

5.10 Lemma. The following propositions are equivalent:
- γ is an expansion in τ;
- $\gamma \approx \{G' \mid \mathrm{Lim}(G') \cap \Upsilon_\tau(\gamma \cup \{G'\}) = \emptyset\}$;
- $\gamma \approx \{G' \mid \mathrm{Lim}(G') \cap \mathrm{Th}_{\mathrm{LPD}}(\Upsilon_\tau(\gamma) \cup \mathrm{Hyp}(G')) = \emptyset\}$.

5.11 Lemma. If $\gamma \approx \gamma'$, then $\Upsilon_\tau(\gamma) = \Upsilon_\tau(\gamma')$.

5.12 Lemma. If $G' \preceq \gamma$, then $\mathrm{Hyp}(G') \subseteq \Upsilon_\tau(\gamma)$.

5.13 Lemma. $\Psi_\tau(\Phi) = \Upsilon_\tau(\overline{\Psi}_\tau(\Phi))$.

The next theorem is an immediate consequence of lemma 5.13.

5.14 Theorem.
- E is an extension in τ if, and only if, $\Upsilon_\tau(\overline{\Psi}_\tau(E)) = E$.

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[25] "$\mathrm{Hyp}(\gamma)$" is read "*hypotheses of* γ".
[26] "$\mathrm{Lim}(\gamma)$" is read "*limits of* γ".
[27] Some trouble can occur if the context is not taken into account with respect to the use of the operators "Hyp" and "Lim". If the empty set is considered a rule, then $\mathrm{Hyp}(\emptyset) = \{\top?\}$ and $\mathrm{Lim}(\emptyset) = \{\bot?\}$, whereas $\mathrm{Hyp}(\emptyset) = \mathrm{Lim}(\emptyset) = \emptyset$, if the empty set is a candidate.

The following theorems state some correspondences between extensions and expansions in an LPR-basis τ.

5.15 Theorem.

- If γ is an expansion in τ, then $\Upsilon_\tau(\gamma)$ is an extension in τ.

Proof:.
(a) We want to show that $\Psi_\tau(\Upsilon_\tau(\gamma)) \subseteq \Upsilon_\tau(\gamma)$.
As $\Psi_\tau(\Upsilon_\tau(\gamma))$ is the least set having properties (i), (ii) and (iii) of definition 4.16, it is enough to show that $\Upsilon_\tau(\gamma)$ satisfies these conditions. As a matter of fact, conditions (i) and (ii) of definition 4.16 follow directly from definition 5.7 of $\Upsilon_\tau(\gamma)$, for any candidate γ.
If

$$\mathrm{Lim}(G') \cap \mathrm{Th}_{\mathrm{LPD}}(\Upsilon_\tau(\gamma) \cup \mathrm{Hyp}(G')) = \emptyset,$$

then, by lemma 5.10, $G' \preceq \gamma$, therefore, by lemma 5.12,

$$\mathrm{Hyp}(G') \subseteq \Upsilon_\tau(\gamma).$$

(b) We want to show that $\Upsilon_\tau(\gamma) \subseteq \Psi_\tau(\Upsilon_\tau(\gamma))$.
Note that, by definitions 4.16 and 5.7, it is enough to show that

$$\mathrm{Hyp}(\gamma) = \bigcup_{G' \in \gamma} \mathrm{Hyp}(G') \subseteq \Psi_\tau(\Upsilon_\tau(\gamma)).$$

If $G' \in \gamma$, by lemma 5.10,

$$\mathrm{Lim}(G') \cap \mathrm{Th}_{\mathrm{LPD}}(\Upsilon_\tau(\gamma) \cup \mathrm{Hyp}(G')) = \emptyset,$$

then, by condition (iii) of definition 4.16,

$$\mathrm{Hyp}(G') \subseteq \Psi_\tau(\Upsilon_\tau(\gamma)). \quad \square$$

The converse of theorem 5.15 does not hold.

5.16 Example. Let $T = \emptyset$ and $G = \{\, p \multimap\!\!(, q \multimap\!\!(, p \wedge q \multimap\!\!(\,\}$.
If $\gamma = \{\, \{\, p \multimap\!\!(, q \multimap\!\!(\,\} \,\}$, then $\Upsilon_\tau(\gamma)$ is an extension in τ, but γ is not an expansion in τ.

5.17 Theorem.

- If E is an extension in τ, then $\overline{\Psi}_\tau(E)$ is an expansion in τ.

Proof:.
According to scholium 4.17,

$$\overline{\Psi}_\tau(E) = \{G' \mid \mathrm{Lim}(G') \cap \mathrm{Th}_{\mathrm{LPD}}(E \cup \mathrm{Hyp}(G')) = \emptyset\}.$$

But, by theorem 5.14, $E = \Upsilon_\tau(\overline{\Psi}_\tau(E))$, so

$$\overline{\Psi}_\tau(E) = \{G' \mid \mathrm{Lim}(G') \cap \mathrm{Th}_{\mathrm{LPD}}(\Upsilon_\tau(\overline{\Psi}_\tau(E)) \cup \mathrm{Hyp}(G')) = \emptyset\},$$

therefore, by lemma 5.10, $\overline{\Psi}_\tau(E)$ is an expansion in τ. $\square$

The converse of theorem 5.17 does not hold, however theorem 5.19 below is proven.

5.18 Example. Let $T = \emptyset$ and $G = \{\, p \prec q,\, q \prec \,\}$.
If $E = \mathrm{Th}_{\mathrm{LPD}}(\{p?,\, q?\})$, then $\overline{\Psi}_\tau(E) = \{\, q \prec \,\}$ is an expansion in τ, but E is not an extension in τ.

5.19 Theorem. If $\overline{\Psi}_\tau(E)$ is an expansion in τ, then $\Psi_\tau(E)$ is an extension in τ.

Proof:. By theorem 5.15, $\Upsilon_\tau(\overline{\Psi}_\tau(E))$ is an extension in τ, hence, by lemma 5.13, $\Psi_\tau(E)$ is an extension in τ. □

5.20 Theorem. The following propositions are equivalent:
- E is an extension in τ;
- there is an expansion γ in τ such that $\Upsilon_\tau(\gamma) = E$.

Proof:.
If E is an extension in τ, we have, by theorem 5.14, that $\Upsilon_\tau(\overline{\Psi}(E)) = E$, but $\overline{\Psi}(E)$ is an expansion, according to theorem 5.17.
If there is an expansion γ in τ such that $\Upsilon_\tau(\gamma) = E$, then, according to theorem 5.15, E is an extension in τ. □

A candidate γ is an expansion in τ if, and only if, γ is a fixed point of the operator $\overline{\Psi}_\tau \circ \Upsilon_\tau$, up to equivalence of candidates.

5.21 Theorem.
- γ is an expansion in τ if, and only if, $\overline{\Psi}_\tau(\Upsilon_\tau(\gamma)) \approx \gamma$.

Proof:.
According to scholium 4.17,
$$\overline{\Psi}_\tau(\Upsilon_\tau(\gamma)) = \{\, G' \mid \mathrm{Lim}(G') \cap \mathrm{Th}_{\mathrm{LPD}}(\Upsilon_\tau(\gamma) \cup \mathrm{Hyp}(G')) = \emptyset \,\}.$$
But, by lemma 5.10, γ is an expansion in τ if, and only if,
$$\gamma \approx \{\, G' \mid \mathrm{Lim}(G') \cap \mathrm{Th}_{\mathrm{LPD}}(\Upsilon_\tau(\gamma) \cup \mathrm{Hyp}(G')) = \emptyset \,\}. \quad \square$$

5.22 Theorem. The following propositions are equivalent:
- γ is an expansion in τ;
- there is an extension E in τ such that $\overline{\Psi}_\tau(E) \approx \gamma$.

Proof:.
If γ is an expansion in τ, then, by theorem 5.15, $\Upsilon_\tau(\gamma)$ is an extension in τ, but, by theorem 5.21, $\overline{\Psi}_\tau(\Upsilon_\tau(\gamma)) \approx \gamma$.
Conversely, if there is an extension E in τ such that $\overline{\Psi}_\tau(E) \approx \gamma$, then, according to theorem 5.17, $\overline{\Psi}_\tau(E)$ is an expansion in τ, therefore, by definition 5.9 and lemma 5.11, γ is an expansion in τ. □

5.23 Theorem. The following propositions are equivalent:
- T is LPD-trivial;
- τ has only one expansion that is equal to $\emptyset$.

5.24 Definition. A *candidate* γ *in* τ is said to be *essential in* τ if the following condition is satisfied:
- for all $G', G'' \in \gamma$, $G' \preceq G''$ implies that $G' = G''$.

All rules of an essential candidate are *maximal*.

5.25 Lemma.
- For each candidate γ in τ, there is an essential candidate γ' in τ such that $\gamma \approx \gamma'$.

The following results apply for well behaved theories, the ones with only one extension in τ, also called, hereafter, *uni-extensional*.

5.26 Theorem. If τ has only one extension and γ is an essential expansion in τ, then the following propositions are valid:
- for each expansion γ' in τ, $\gamma \approx \gamma'$;
- if $G' \in \gamma$, then $T \cup \mathrm{Conj}(G')$ is a plausible scenario in τ;
- if $P \in \mathrm{Conj}(G')$, for some $G' \in \gamma$, then $P?$ is a plausible formula in τ[28];
- if $P \in Conj(G')$, for all $G' \in \gamma$, then $P!$ is a strictly plausible formula in τ[29];
- $\Pi(\tau) = \mathrm{Th}_{\mathrm{LPD}}(T \cup T_1 \cup T_2 \cup T_3)$, whereon:
  - $T_1 = \{ \Diamond P \mid P$ is a modality-free sentence and $T \cup \{P\}$ is not LPD-trivial $\}$;
  - $T_2 = \{ (\bigwedge \mathrm{Conj}(G'))? \mid G'$ is finite and G' is subset of some element of $\gamma \}$;
  - $T_3 = \{ P! \mid P \in \mathrm{Conj}(G'),$ for all $G' \in \gamma \}$.

For general theories, a triggered collection of generalizations can be defined as follows:

5.27 Theorem.
- G' is triggered in τ if, and only if, $G' \preceq \gamma$, for all expansions γ in τ.

At the same way as in section 4, plausible scenarios and the set of theorems of τ can be characterized using expansions.

Next, we present yet another way for characterizing expansions in τ.

5.28 Definition. The following clauses specify some relations between rules and candidates in τ:
- G' *rejects* G'' *in* τ $\rightleftharpoons$ there is a finite $G_0 \subseteq G''$ such that $T \cup \mathrm{Conj}(G') \vdash_{\mathrm{CL}} \bigvee \mathrm{Rest}(G_0)$ or $T \cup \mathrm{Conj}(G'') \vdash_{\mathrm{CL}} \bigvee \mathrm{Rest}(G_0)$;
- γ *rejects* G' *in* τ $\rightleftharpoons$ there exists $G_0 \in \gamma$ such that G_0 rejects G' in τ, or G' rejects G' in τ;
- G' *rejects* γ *in* τ $\rightleftharpoons$ there exists $G_0 \in \gamma$ such that G' rejects G_0 in τ.

5.29 Scholium. The following propositions are equivalent:
- G' rejects G' in τ;
- there is a finite $G_0 \subseteq G'$ such that $T \cup \mathrm{Conj}(G') \vdash_{\mathrm{CL}} \bigvee \mathrm{Rest}(G_0)$.

5.30 Scholium. The following propositions are equivalent:
- G' rejects G'' in τ;
- there is a finite $G_0 \subseteq G''$ such that $T \cup \mathrm{Conj}(G') \vdash_{\mathrm{CL}} \bigvee \mathrm{Rest}(G_0)$ or G'' rejects G'' in τ.

[28] That is, $P?$ is theorem of τ.
[29] That is, $P!$ is theorem of τ.

5.31 Scholium. The following propositions are equivalent:
- γ rejects G' in τ;
- there is $G_0 \in \gamma$ and there is a finite $G_1 \subseteq G'$ such that $T \cup \mathrm{Conj}(G_0) \mathrel{|\!\underline{\;\;}_{\mathrm{CL}}} \bigvee \mathrm{Rest}(G_1)$ or $T \cup \mathrm{Conj}(G') \mathrel{|\!\underline{\;\;}_{\mathrm{CL}}} \bigvee \mathrm{Rest}(G_1)$.

5.32 Lemma. The following propositions are valid:
- if $\begin{cases} G_1 \text{ rejects } G_2 \text{ in } \tau, \\ G_1 \preceq G_1', \\ G_2 \preceq G_2', \end{cases}$ then G_1' rejects G_2' in τ;
- if $\begin{cases} \gamma \text{ rejects } G' \text{ in } \tau, \\ \gamma \preceq \gamma', \\ G' \preceq G'', \end{cases}$ then γ' rejects G'' in τ;
- if $\begin{cases} G' \text{ rejects } \gamma \text{ in } \tau, \\ G' \preceq G'', \\ \gamma \preceq \gamma', \end{cases}$ then G'' rejects γ' in τ.

5.33 Lemma. The following propositions are equivalent:
- $\mathrm{Lim}(G') \cap \Upsilon_\tau(\gamma) \neq \emptyset$;
- there is $G_0 \in \gamma$ and there is a finite $G_1 \subseteq G'$ such that $T \cup \mathrm{Conj}(G_0) \mathrel{|\!\underline{\;\;}_{\mathrm{CL}}} \bigvee \mathrm{Rest}(G_1)$.

Proof:.
$\mathrm{Lim}(G') \cap \Upsilon_\tau(\gamma) \neq \emptyset$

if, and only if (according to definitions 4.13 and 5.7),

there is a finite $G_1 \subseteq G'$ such that $T \cup \mathrm{Hyp}(\gamma) \mathrel{|\!\underline{\;\;}_{\mathrm{LPD}}} (\bigvee \mathrm{Rest}(G_1))?$

if, and only if (by lemma 3.16),

there is $G_0 \in \gamma$ and there is a finite $G_1 \subseteq G'$ and there is a finite $G_2 \subseteq G_0$ such that $T \cup (\bigwedge \mathrm{Conj}(G_2))? \mathrel{|\!\underline{\;\;}_{\mathrm{LPD}}} (\bigvee \mathrm{Rest}(G_1))?$

if, and only if (again by lemma 3.16),

there is $G_0 \in \gamma$ and there is a finite $G_1 \subseteq G'$ and there is a finite $G_2 \subseteq G_0$ such that $T \cup \bigwedge \mathrm{Conj}(G_2) \mathrel{|\!\underline{\;\;}_{\mathrm{CL}}} \bigvee \mathrm{Rest}(G_1)$

if, and only if (by lemma 4.11),

there is $G_0 \in \gamma$ and there is a finite $G_1 \subseteq G'$ and there is a finite $G_2 \subseteq G_0$ such that $T \cup \mathrm{Conj}(G_2) \mathrel{|\!\underline{\;\;}_{\mathrm{CL}}} \bigvee \mathrm{Rest}(G_1)$

if, and only if (by compactness of axiomatic calculi),

there is $G_0 \in \gamma$ and there is a finite $G_1 \subseteq G'$ such that $T \cup \mathrm{Conj}(G_0) \mathrel{|\!\underline{\;\;}_{\mathrm{CL}}} \bigvee \mathrm{Rest}(G_1)$. □

5.34 Lemma. The following propositions are equivalent:
- γ rejects G' in τ;
- $\mathrm{Lim}(G') \cap \Upsilon_\tau(\gamma \cup \{G'\}) \neq \emptyset$.

Proof:.
γ rejects G' in τ

if, and only if (according to scholium 5.31), there is $G_0 \in \gamma$ and there is a finite $G_1 \subseteq G'$ such that $T \cup \mathrm{Conj}(G_0) \mathrel{|}\!\!\frac{}{\mathrm{CL}} \bigvee \mathrm{Rest}(G_1)$ or $T \cup \mathrm{Conj}(G') \mathrel{|}\!\!\frac{}{\mathrm{CL}} \bigvee \mathrm{Rest}(G_1)$

if, and only if (by lemma 5.33),

$\mathrm{Lim}(G') \cap \Upsilon_\tau(\gamma) \neq \emptyset$ or $\mathrm{Lim}(G') \cap \Upsilon_\tau(\{G'\}) \neq \emptyset$

if, and only if,

$\mathrm{Lim}(G') \cap \Upsilon_\tau(\gamma \cup \{G'\}) \neq \emptyset$. $\square$

5.35 Theorem. The following propositions are equivalent:
- γ is an expansion in τ;
- for each G', $G' \preceq \gamma$ if, and only if, γ does not reject G' in τ.

Proof:. It follows directly from definition 5.9 and lemma 5.34. $\square$

5.36 Definition. The following clauses define some qualities related to candidates in τ:
- γ is *sound in* τ $\rightleftharpoons$ for each $G' \in \gamma$, γ does not reject G' in τ;
- γ is *complete in* τ $\rightleftharpoons$ for each G', if γ does not reject G' in τ, then $G' \preceq \gamma$;
- γ is *full in* τ $\rightleftharpoons$ for each G', if $\gamma \cup \{G'\}$ is sound in τ, then $G' \preceq \gamma$;
- γ *complies with the exceptions-first criterion in* τ $\rightleftharpoons$ for each G', if G' rejects γ in τ, then γ rejects G' in τ.

5.37 Lemma. The following propositions are equivalent:
- γ is sound in τ;
- for each $G' \preceq \gamma$, γ does not reject G' in τ.

5.38 Lemma.
- If γ is sound in τ and $G' \preceq \gamma$, then G' does not reject γ in τ.

5.39 Lemma. The following propositions are equivalent:
- $\gamma \cup \{G'\}$ is sound in τ;
- γ is sound in τ, γ does not reject G' in τ and G' does not reject γ in τ.

5.40 Theorem. The following propositions are equivalent:
- γ is an expansion in τ;
- γ is sound and complete in τ.

Proof:.
Suppose that γ is an expansion in τ.
If $G' \in \gamma$, then $G' \preceq \gamma$, so, by theorem 5.35, γ does not reject G' in τ, therefore γ is sound in τ.
If γ does not reject G' in τ, then, again by theorem 5.35, $G' \preceq \gamma$, therefore γ is complete in τ.

Now suppose that γ is sound and complete in τ.
If $G' \preceq \gamma$, then, considering that γ is sound in τ and lemma 5.37, γ does not reject G' in τ.
If γ does not reject G' in τ, then, as γ is complete in τ, we have that $G' \preceq \gamma$.
Therefore, by theorem 5.35, γ is an expansion in τ. □

5.41 Lemma. If γ is complete in τ, then γ is full in τ.

5.42 Example. Let $T = \emptyset$ and $G = \{\, p \prec p,\ q \prec \,\}$.
If $\gamma = \{\, \{\, p \prec p \,\} \,\}$, then γ is full in τ, but γ is not complete in τ, nor it is sound in τ.

5.43 Theorem. The following propositions are equivalent:
- γ is an expansion in τ;
- γ is sound, full and complies with the exceptions-first criterion in τ.

Proof:.
Suppose that γ is an expansion in τ.
By theorem 5.40, we have that γ is sound and complete in τ, so, by lemma 5.41, γ is full in τ. It remains to demonstrate that γ complies with the exceptions-first criterion in τ.
If G' rejects γ in τ, then, considering that γ is sound in τ and lemma 5.38, $G' \npreceq \gamma$, so, by theorem 5.35, γ rejects G' in τ, therefore γ complies with the exceptions-first criterion in τ.

Now suppose that γ is sound, full and complies with the exceptions-first criterion in τ.
If $G' \preceq \gamma$, then, considering that γ is sound in τ and lemma 5.37, we have that γ does not reject G' in τ.
If γ does not reject G' in τ, then, as γ complies with the exceptions-first criterion in τ, G' does not reject γ in τ, hence, as γ is sound in τ, we have by lemma 5.39 that $\gamma \cup \{G'\}$ is sound in τ, so, as γ is full in τ, $G' \preceq \gamma$.
Therefore, according to theorem 5.35, γ is an expansion in τ. □

5.44 Example (revision of example 4.33)**.**
Consider the LPR-basis $\tau = \langle T, G \rangle$ of example 4.33:

$$T = \{\ \mathrm{bird(Tweety)}, \mathrm{bird(Woody)}\ \};$$

$$\begin{aligned} G = \{\ & (\mathrm{bird}(x) \to \mathrm{flies}(x)) \prec \mathrm{penguin}(x)\,, \\ & (\mathrm{penguin}(x) \to \neg\, \mathrm{flies}(x)) \prec\,, \\ & \mathrm{penguin(Tweety)} \prec \,\}\,. \end{aligned}$$

$$\begin{aligned} \gamma = \Big\{ \{\ & (\mathrm{bird}(x) \to \mathrm{flies}(x)) \prec \mathrm{penguin}(x)\,, \\ & (\mathrm{penguin}(x) \to \neg\, \mathrm{flies}(x)) \prec \,\} \Big\} \end{aligned}$$

is not an expansion in τ for it does not conform to the exceptions first criterion. Note that "penguin(Tweety) $\prec$" rejects γ, but γ does not reject it. We have that $\Upsilon_\tau(\gamma)$, which equals $\mathrm{Th}_{\mathrm{LPD}}(T \cup \Phi)$, whereon

$$\Phi = \left\{ \left(\forall x \left((\mathrm{bird}(x) \to \mathrm{flies}(x)) \wedge (\mathrm{penguin}(x) \to \neg\, \mathrm{flies}(x))\right)\right)? \right\},$$

is not an extension in τ, so the proposition "$\neg\,$penguin(Tweety)?" is not plausible in τ.

6. Conclusions

In this paper we presented a logic to express reasoning. By this term it is meant a wide variety of inferential practices, with two main characteristics: it allows the derivation of conclusions that does not necessarily preserve the truth of the premises. It embodies in the analysis statements describing alternative scenarios, possibly even ones that contradict each other. Those features play an essential role when real life, practical, effective reasoning is concerned. The first feature, the non conservativeness of truth, characterizes it as *ampliative reasoning*; the second, the consideration of alternative plausible possibilities as premises submitted to analysis makes of it *complex reasoning*.

These characteristics present challenges to the systematization of their treatment and technical problems to their logical formalization. The work we faced on was precisely to offer a solution to meet these challenges, and its result is here presented in the form of a logic able to express relevant features of a large class of complex ampliative reasoning, if we have hopefully been well succeeded.

In the process of developing this solution we made some choices and took some methodological decisions. One of them was to compromise with a qualitative logic-like approach. This has some clear advantages in comparison to probabilistic treatment, for instance, but also some drawbacks. Notice that we characterize as (weakly) plausible something occurring in at least one scenario, but we have no means to distinguishing among something occurring in just one scenario from some other thing which occurs in all but one scenario. We don't count, and that is the price we pay. This does not mean, however, that what we get is not relevant, for the consideration of some catastrophic possible occurrence in the worst of the plausible scenarios is a matter of interest in any sensible analysis. On the other hand, it may happen, and usually does happen, that the competing conjectures allowing alternative scenarios entail conclusions occurring in all scenarios. Those are occurrences of strong plausibility, and they go beyond the ones that can be inferred just by considering the assured knowledge. Those conclusions can be taken as *pragmatic truths*, points of consensus among concurrent conjectures or theories, which may be considered as certainties for practical concern.

Another important feature of our work is this idea of treating conjectures rounding out knowledge which is taken for granted, and thus being able to reason taking into account all alternative scenarios as composing a same framework. Of course the logic does not help to raise the right conjectures that give birth to the framework, for this is an empirical task that cannot be performed by logic (unless it is a logic of discovering, something the authors do not too much trust in), but, once those conjectures are provided, LPR furnish the means to construct the alternative possible and plausible scenarios, and to draw the logical consequences upon them, in the same way classical logic does from the premises provided.

Complex reasoning prevails everywhere. In all situations in which knowledge needs to be rounded out with putative guesses and conjectures, complex reasoning comes into play. In each matter of consideration there always are different points of view — hopefully not too many — to account for. This is the case for practical reasoning in everyday life, nonmonotonic reasoning in Artificial Intelligence, and even scientific reasoning, to name just a few. Sometimes there are assertions completely justified on their own, and yet, they clash when put together. This happens when inconsistencies are found in Physics whilst considering its frontiers, and it is apparent in the early stages of development of scientific disciplines, such as the so called cognitive sciences nowadays. It is also the case for ancient but complex disciplines. Ethics, where moral dilemmas often flourish, is one example. Life is too complex and there is no such a thing as "the truth" or "the right thing to do".

In advancing LPR, a logic constructed accordingly with strict mathematical methods, we hope we have climbed one step towards the understanding and formalization of reasoning in all its richness of aspects.

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