A General Treatment for the Deduction Theorem in Open Calculi

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Abstract

Aiming clear formulations of the deduction theorem and precise account of its restrictions, a study concerning the rules introducing connectives relative to its behavior is conducted. Special attention is devoted to operators, such as quantifiers and modalities, dealing with varying objects. Concepts and techniques able to cope with the subtleties of tracing those objects are introduced. As a result, a classification of calculi in terms of robustness to applications of the deduction theorem is given.

1 Introduction

The material implication, no matter of having always being one of the most disputed connectives along all the history of Logic, is almost universally present in the myriad of logics nowadays available. In addition to this popularity, it plays a unique role, in most of the logics it occurs, because it has a direct connection to the very core of the particular relation of logical consequence that the logic intends to define. In fact, it works as a formulation for that notion, expressed in terms of the internal logical formalism. The relation between this internal notion of entailment it represents and the notion of logical consequence the logical calculus as a hole express is usually formulated through the so-called deduction theorem, which establishes the ways and conditions for their interplay. In general, the way the implication is introduced in a particular calculus is related to the way the rules for quantifiers and other connectives alike, such as modalities, are introduced.

In spite of being such a pervasive connective and a so important one, it can be verified that frequently it is not treated in a way so careful as it deserves, bringing about imprecision in the enunciation of rules and misapplications of theorems related to its definition. The aim of the authors in this paper is precisely the one of clearing out the issue, by making explicit all the elements and relations involved in the formulations of the rules for the introduction of some key connectives that have the power to interfere in the correct enunciation of the deduction theorem applicable in a particular calculus. Among these connectives there are quantifiers and other operators of the kind, such as modalities, whose occurrence in a formula may require some sort of tracing of the varying objects that they, whether explicitly or implicitly, may involve, and that may affect the behavior of the terms occurring in the deduction theorem. This concept of a varying object, which generalizes the one of a variable, together with two basic relations, named here as dependency and supporting, to be introduced later in the paper, are the key elements in our strategy to make explicit the mutual relations among the rules introducing those connectives and the way they affect the correct enunciation and applications of the deduction theorem. The treatment given here for the subject is conducted in the style of generalized logic, meaning the introducing of concepts, theorems and procedures with no special regards for particular systems but, instead of it, using them to characterize classes of logics with respect to properties related to these concepts. It turns out, from this strategy, the results being applicable to large classes of logics, whether known or yet to develop. The concepts here presented had been already applied by the authors in the formulations and generalized proofs of metatheorems for some non classical calculi in [2] and [3].

It can be observed by a careful examination of the classical logic books that two distinct choices for the introduction of implication and quantifiers have been made standard:

1st) The rule for introducing the implication has no restrictions, but there are constraints for introducing the universal quantifier and other operators of the kind. A calculus adopting this strategy is called closed in our context. It is more often used in calculi presented in natural deduction and sequent calculus style. Examples of closed calculi may be found in [1], [10], [4], [5] and [8]. However, this closed option may be very cumbersome when used to calculi presented in axiomatic style having varying objects other than variables, such as in modal logics.

2nd) The introduction of implication is done with restrictions, but the introduction of the universal quantifier and other analogous operators is unconditional. This strategy is more often adopted for axiomatic formulations. From here on, this kind of calculi are called open. Examples may be found in [6], [7] and [9].

It is important to notice that these alternative strategies are not incompatible and have been adopted as variants for the definition of the same logic, classical logic, for instance. However, the resulting calculi are not strictly equivalent. They are equivalent just in the sense of having the same logical theorems but not in the stronger sense of implementing the same deduction relation. For instance, in an open formalization for the classical logic we have $p(x) \vdash \forall x\, p(x)$, but the same does not hold for a closed formalization.

Some well known formulations of the deduction theorem, in the context of open calculi, presented in axiomatic style, that can be found in the literature, present some undesirable features, such as:

- explicit use of the concept of demonstration, instead of an idea of a higher level dealing with syntactic consequence;
- lack of an adequate tracing to accompany the use of varying objects in rules of generalization, making it difficult further applications of the deduction theorem, in a context, after the first time it has been applied.

Furthermore, we consider it essential, for a deeper understanding of this matter, to conduct a survey, followed by a careful study, of the basic properties of the consequence relations involved. We have discovered in this study two relevant relations of consequence, with different tracing systems for varying objects, here called *dependence* and *supporting*. Under certain special conditions, to be precisely stated later, these relations can be proved to be equivalent, enabling the use of the most convenient properties of both of them.

Below we will give two examples of formulations of the deduction theorem, commonly found in the literature, that suffer from the above-mentioned ills:

- "For the predicate calculus (or the full number-theoretic formal system), if $\Gamma, A \vdash B$ with the free variables held constant for the last assumption formula A, then $\Gamma \vdash A \to B$." According to [6], p. 97.
- "Assume that $\Gamma, A \vdash B$, where, in the deduction, no application of Gen to a wf which depends upon A has as its quantified variable a free variable of A. Then $\Gamma \vdash A \to B$." According to [7], p. 63.

In this study, we have found formulations for the deduction theorem that overcome these problems in a generalization that covers a broad spectrum of logics.

2 Variation, Dependence and Supporting

From now on, we will consider **C** an axiomatic calculus, α,β,γ formulas in **C**, and Γ,Φ,ζ collections of formulas in **C**; the same conventions continue to be valid if the cited signals are used with primes and/or subscripts.

2.1 Pre-Definition. For each application of a rule of inference **r** in **C**, we consider as previously known *the varying objects of this application in* **C**. If **r** is a rule in **C**, whose applications do not have varying objects, we say that **r** is a *constant rule in* **C**; otherwise we say that **r** is a *varying rule in* **C**. We say that **o** is a *varying object in* **C** if there is an application of a rule in **C** such that **o** is a varying object of this application. We also consider as previously known when a varying object **o** is *free in a given formula* α. The following additional conditions are to be fulfilled:

- the number of varying objects of each application of a rule in **C** is finite;
- each varying object of an application of a rule is not free in the consequence of this application.

2.2 Examples. In practice, we find the following varying objects:

- variables used in universal quantification: "x" is the varying object of the application $\frac{\alpha}{\forall x\, \alpha}$ of the rule of universal generalization, which occurs in many quantificational logics;
- the hidden variable used for introducing connectives associated with modalities such as necessity; such variable can be indicated by the sign itself introduced by the rule: "$\Box$" is the varying object of the rule $\frac{\alpha}{\Box\alpha}$, which occurs in many modal logics.

2.3 Definition. Let $\mathcal{D} = \alpha_1, \ldots, \alpha_n$ be a demonstration in **C**. We say that α_i *is relevant to* α_j *in* $\mathcal{D}$ $(i, j \in \{1, \ldots, n\})$ if one of the following conditions is fulfilled:

- $i = j$ and α_j is justified in $\mathcal{D}$ as a premise;
- α_j is justified in $\mathcal{D}$ as a consequence of an application $\frac{\alpha_j}{\beta_1,\ldots,\beta_p}$ of a rule in **C** and there exists a hypothesis β_k $(k \in \{1, \ldots, p\})$ of this application such that α_i is relevant to β_k in $\mathcal{D}$.

2.4 Definition. We say that a demonstration $\mathcal{D}$ in $\mathbf{C}$ *depends on* a collection $\mathcal{V}$ of varying objects if $\mathcal{V}$ contains the collection of varying objects $\mathbf{o}$ of applications of rules in $\mathcal{D}$ having a hypothesis in which $\mathbf{o}$ is free such that there is a formula, justified as a premise in $\mathcal{D}$, whereon $\mathbf{o}$ is free too, relevant to this hypothesis in $\mathcal{D}$. If there is a demonstration in $\mathbf{C}$ of α from Γ such that it depends on $\mathcal{V}$, we say that α depends on $\mathcal{V}$ from Γ in $\mathbf{C}$, and we note this by $\Gamma \mathrel{\vdash^{\mathcal{V}}_{\mathbf{C}}} \alpha$. If $\mathcal{V} = \{\mathbf{o}_1, \ldots, \mathbf{o}_n\}$ and $n \geq 1$, we also note this by $\Gamma \mathrel{\vdash^{\mathbf{o}_1,\ldots,\mathbf{o}_n}_{\mathbf{C}}} \alpha$. If $\mathcal{V} = \emptyset$, we say that $\mathcal{D}$ is an *unvarying demonstration in* $\mathbf{C}$. If α depends on $\emptyset$ from Γ in $\mathbf{C}$, we say that it is an *unvarying consequence of* Γ *in* $\mathbf{C}$.

2.5 Theorem. A formula α depends on $\mathcal{V}$ from Γ in $\mathbf{C}$ if, and only if, at least one of the following conditions is fulfilled:

- α is an axiom of $\mathbf{C}$;
- $\alpha \in \Gamma$;
- there is an application $\frac{\alpha_1,\ldots,\alpha_n}{\alpha}$ of a rule in $\mathbf{C}$ such that $\Gamma \mathrel{\vdash^{\mathcal{V}}_{\mathbf{C}}} \alpha_1, \ldots, \Gamma \mathrel{\vdash^{\mathcal{V}}_{\mathbf{C}}} \alpha_n$ and, for every varying object $\mathbf{o}$ of this application such that $\mathbf{o} \notin \mathcal{V}$ and for every α_i $(1 \leq i \leq n)$, if $\mathbf{o}$ is free in α_i, then there is $\Gamma' \subseteq \Gamma$, such that $\mathbf{o}$ is not free in Γ' and $\Gamma' \mathrel{\vdash^{\mathcal{V}}_{\mathbf{C}}} \alpha_i$.

If $\mathcal{V} = \emptyset$, we can replace the third clause by the following condition:

- there exists an application $\frac{\alpha_1,\ldots,\alpha_n}{\alpha}$ of a rule in $\mathbf{C}$ such that $\Gamma \mathrel{\vdash^{\emptyset}_{\mathbf{C}}} \alpha_1, \ldots, \Gamma \mathrel{\vdash^{\emptyset}_{\mathbf{C}}} \alpha_n$ and, for every varying object $\mathbf{o}$ of this application and for every α_i $(1 \leqslant i \leqslant n)$, if $\mathbf{o}$ is free in α_i, then there exists $\Gamma' \subseteq \Gamma$ such that $\mathbf{o}$ is not free in Γ' and $\Gamma' \mathrel{\vdash^{\emptyset}_{\mathbf{C}}} \alpha_i$.

2.6 Examples. In an open axiomatic calculus with the rules of universal generalization and of necessity, we have the following examples of dependence:

- $p(x, y) \mathrel{\vdash^{x,y}} \forall x \forall y\, p(x, y)$;
- $px \mathrel{\vdash^{x,\Box}} \Box \forall x\, px$.

2.7 Theorem. The following properties are valid for the relation "$\vdash^{\mathcal{V}}_{\mathbf{C}}$":

(i) if there is a demonstration $\mathcal{D}$ in $\mathbf{C}$ of α from Γ whose collection of varying objects of applications of rules of $\mathbf{C}$ in $\mathcal{D}$ is $\mathcal{V}$, then $\Gamma \mathrel{\vdash^{\mathcal{V}}_{\mathbf{C}}} \alpha$;

(ii) if $\Gamma \mathrel{\vdash^{\mathcal{V}}_{\mathbf{C}}} \alpha$, then $\Gamma \mathrel{\vdash_{\mathbf{C}}} \alpha$;

(iii) if $\Gamma \mathrel{\vdash_{\mathbf{C}}} \alpha$, then there is a collection $\mathcal{V}$ of varying objects such that $\Gamma \mathrel{\vdash^{\mathcal{V}}_{\mathbf{C}}} \alpha$;

(iv) $\vdash_{\mathbf{C}} \alpha$ iff $\vdash^{\emptyset}_{\mathbf{C}} \alpha$;

(v) if $\Gamma \mathrel{\vdash^{\mathcal{V}}_{\mathbf{C}}} \alpha$ and $\mathcal{V} \subseteqq \mathcal{V}'$, then $\Gamma \mathrel{\vdash^{\mathcal{V}'}_{\mathbf{C}}} \alpha$;

(vi) if $\Gamma \mathrel{\big|\!\frac{\mathcal{V}}{\mathbf{C}}} \alpha$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \mathrel{\big|\!\frac{\mathcal{V}}{\mathbf{C}}} \alpha$;

(vii) if $\Gamma \mathrel{\big|\!\frac{\mathcal{V}}{\mathbf{C}}} \alpha$, then there is $\mathcal{V}' \subseteq \mathcal{V}$ such that $\mathcal{V}'$ is finite and $\Gamma \mathrel{\big|\!\frac{\mathcal{V}'}{\mathbf{C}}} \alpha$;

(viii) if $\Gamma \mathrel{\big|\!\frac{\mathcal{V}}{\mathbf{C}}} \alpha$, then there is $\Gamma' \subseteq \Gamma$ such that Γ' is finite and $\Gamma' \mathrel{\big|\!\frac{\mathcal{V}}{\mathbf{C}}} \alpha$;

(ix) if $\Gamma \mathrel{\big|\!\frac{\mathcal{V}}{\mathbf{C}}} \alpha$ and, for each $\mathbf{o} \in \mathcal{W}$, $\mathbf{o}$ is not free in Γ, then $\Gamma \mathrel{\big|\!\frac{\mathcal{V}-\mathcal{W}}{\mathbf{C}}} \alpha$;

(x) if $\begin{cases} * \ \Gamma \mathrel{\big|\!\frac{\mathcal{V}}{\mathbf{C}}} \alpha, \\ * \ \text{for each } \mathbf{o} \in \mathcal{W}, \text{ there exists } \Gamma' \subseteq \Gamma \text{ such that } \mathbf{o} \text{ is not free} \\ \quad \text{in } \Gamma' \text{ and } \Gamma' \mathrel{\big|\!\frac{\mathcal{V}}{\mathbf{C}}} \alpha, \end{cases}$

then $\Gamma \mathrel{\big|\!\frac{\mathcal{V}-\mathcal{W}}{\mathbf{C}}} \alpha$.

2.8 Example. The following assertions are not valid for the relation "$\big|\!\frac{\mathcal{V}}{\mathbf{C}}$":

- if $\Gamma \mathrel{\big|\!\frac{\mathcal{V}}{\mathbf{C}}} \alpha_1, \ldots, \Gamma \mathrel{\big|\!\frac{\mathcal{V}}{\mathbf{C}}} \alpha_n, \{\alpha_1, \ldots, \alpha_n\} \mathrel{\big|\!\frac{\mathcal{W}}{\mathbf{C}}} \beta$, then $\Gamma \mathrel{\big|\!\frac{\mathcal{V}_1 \cup \cdots \cup \mathcal{V}_n \cup \mathcal{W}}{\mathbf{C}}} \beta$;
- if $\begin{cases} * \ \Gamma \mathrel{\big|\!\frac{\mathcal{V}}{\mathbf{C}}} \alpha_1, \ldots, \Gamma \mathrel{\big|\!\frac{\mathcal{V}}{\mathbf{C}}} \alpha_p, \\ * \ \{\alpha_1, \ldots, \alpha_p\} \mathrel{\big|\!\frac{\mathbf{o}_1, \ldots, \mathbf{o}_n}{\mathbf{C}}} \beta, \\ * \ \text{for all } i \in \{1, \ldots, n\} \text{ and for all } j \in \{1, \ldots, p\}, \text{ if } \mathbf{o}_i \notin \mathcal{V} \\ \quad \text{and } \mathbf{o}_i \text{ is free in } \alpha_j, \text{ then there exists } \Gamma' \subseteq \Gamma \text{ such that} \\ \quad \mathbf{o}_i \text{ is not free in } \Gamma' \text{ and } \Gamma' \mathrel{\big|\!\frac{\mathcal{V}}{\mathbf{C}}} \alpha_j, \end{cases}$

  then $\Gamma \mathrel{\big|\!\frac{\mathcal{V}}{\mathbf{C}}} \beta$.

Proof. Let $\mathbf{C}$ be a calculus whose axioms are given by the schemas "$\alpha \to \alpha \vee \beta$" and "$\forall x\, \alpha \to \alpha(x|t)$", and whose rules of inference are $\frac{\alpha,\ \alpha \to \beta}{\beta}$ and $\frac{\alpha}{\forall x\, \alpha}$, such that the first is a constant rule and the second is a varying rule in which the varying object of each application is the corresponding quantified variable.

We have that $\begin{cases} \{\forall y\, Q(y,z),\ Q(y,z) \to Ry\} \mathrel{\big|\!\frac{\emptyset}{\mathbf{C}}} Ry \\ Ry \mathrel{\big|\!\frac{\emptyset}{\mathbf{C}}} \forall z\, (Ry \vee Sz) \end{cases}$, however it is not true that $\{\forall y\, Q(y,z) \to Ry\} \mathrel{\big|\!\frac{\emptyset}{\mathbf{C}}} \forall z\, (Ry \vee Sz)$, from which we have a counterexample for the first proposition.

Likewise, we have that $\begin{cases} \{\forall y\, Q(y,z),\ Q(y,z) \to Ry\} \mathrel{\big|\!\frac{\emptyset}{\mathbf{C}}} Ry \\ Ry \mathrel{\big|\!\frac{y}{\mathbf{C}}} \forall y \forall z\, (Ry \vee Sz) \end{cases}$, nevertheless it is not true that $\{\forall y\, Q(y,z),\ Q(y,z) \to Ry\} \mathrel{\big|\!\frac{\emptyset}{\mathbf{C}}} \forall y \forall z\, (Ry \vee Sz)$, from which we have a counterexample for the second proposition. □

2.9 Definition. We say that a demonstration $\mathcal{D}$ in $\mathbf{C}$ *is supported by* a collection $\mathcal{V}$ of varying objects if $\mathcal{V}$ contains the collection of varying objects of applications of rules in $\mathcal{D}$ such that, for each conclusion of such applications, there exists a premise relevant to it in $\mathcal{D}$. If there exists a demonstration in $\mathbf{C}$ of α from Γ such that $\mathcal{D}$ is supported by $\mathcal{V}$, we say that α *is supported by*

$\mathcal{V}$ *from* Γ *in* $\mathbf{C}$, and we note this by $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$. If $\mathcal{V} = \{\mathbf{o}_1, \ldots, \mathbf{o}_n\}$ and $n \leq 1$, we also note $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$ by $\Gamma \Vdash^{\mathbf{o}_1,\ldots,\mathbf{o}_n}_{\mathbf{C}} \alpha$. If $\mathcal{V} = \emptyset$, we say that $\mathcal{D}$ is a *stable demonstration in* $\mathbf{C}$. If α is supported by $\emptyset$ from Γ in $\mathbf{C}$, we say that α is a *stable consequence of* Γ *in* $\mathbf{C}$.

2.10 Theorem. If $\mathcal{V}$ is a collection of varying objects in $\mathbf{C}$, then α is supported by $\mathcal{V}$ from Γ in $\mathbf{C}$ if, and only if, at least one of the following clauses is fulfilled:

- α is an axiom of $\mathbf{C}$;
- $\alpha \in \Gamma$;
- there exists an application $\frac{\alpha_1,\ldots,\alpha_n}{\alpha}$ of a rule in $\mathbf{C}$ such that $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_1, \ldots, \Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_n$ and, if there is a varying object $\mathbf{o}$ of this application such that $\mathbf{o} \notin \mathcal{V}$, then $\vdash_{\mathbf{C}} \alpha_1, \ldots, \vdash_{\mathbf{C}} \alpha_n$.

2.11 Example. In an open axiomatic calculus with the rules of generalization and necessity, we have the following example of supporting:

- $\Diamond px \Vdash^{\Box,y} \Box\forall y\, \Diamond px$.

2.12 Theorem. The following properties are valid for the relation "$\Vdash^{\mathcal{V}}_{\mathbf{C}}$":

(i) if there exists a demonstration $\mathcal{D}$ in $\mathbf{C}$ of α from Γ whose collection of varying objects of applications of rules of $\mathbf{C}$ in $\mathcal{D}$ is $\mathcal{V}$, then $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$;

(ii) if $\Gamma \vdash_{\mathbf{C}} \alpha$, then there is a collection $\mathcal{V}$ of varying objects such that $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$;

(iii) $\vdash_{\mathbf{C}} \alpha$ iff $\Vdash^{\emptyset}_{\mathbf{C}} \alpha$;

(iv) if $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$ and $\mathcal{V} \subseteq \mathcal{V}'$, then $\Gamma \Vdash^{\mathcal{V}'}_{\mathbf{C}} \alpha$;

(v) if $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$;

(vi) if $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$, then there exists $\mathcal{V}' \subseteq \mathcal{V}$ such that $\mathcal{V}'$ is finite and $\Gamma \Vdash^{\mathcal{V}'}_{\mathbf{C}} \alpha$;

(vii) if $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$, then there exists $\Gamma' \subseteq \Gamma$ such that Γ' is finite and $\Gamma' \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$;

(viii) if $\Gamma \Vdash^{\mathcal{V}_1}_{\mathbf{C}} \alpha_1, \ldots, \Gamma \Vdash^{\mathcal{V}_n}_{\mathbf{C}} \alpha_n, \{\alpha_1, \ldots, \alpha_n\} \Vdash^{\mathcal{W}}_{\mathbf{C}} \beta$,
then $\Gamma \Vdash^{\mathcal{V}_1 \cup \cdots \cup \mathcal{V}_n \cup \mathcal{W}}_{\mathbf{C}} \beta$.

2.13 Theorem. The following proposition describes a way of expansion for the relation "$\vdash^{\mathcal{V}}_{\mathbf{C}}$" in a generic calculus.

- If $\Gamma \vdash^{\mathcal{V}_1}_{\mathbf{C}} \alpha_1, \ldots, \Gamma \vdash^{\mathcal{V}_n}_{\mathbf{C}} \alpha_n, \{\alpha_1, \ldots, \alpha_n\} \Vdash^{\mathcal{W}}_{\mathbf{C}} \beta$, then $\Gamma \vdash^{\mathcal{V}_1 \cup \cdots \cup \mathcal{V}_n \cup \mathcal{W}}_{\mathbf{C}} \beta$.

2.14 Theorem. If $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$, then $\Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$.

Proof. If α is an axiom of $\mathbf{C}$ or $\alpha \in \Gamma$, there is nothing to prove.
Let us suppose then there is an application $\frac{\alpha_1,\ldots,\alpha_n}{\alpha}$ of a rule of $\mathbf{C}$ fulfilling the conditions of theorem 2.10. By induction hypothesis, we have that $\Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_1, \ldots, \Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_n$. Given a varying object $\mathbf{o}$ of this application such that $\mathbf{o} \notin \mathcal{V}$, we have $\vdash_{\mathbf{C}} \alpha_1, \ldots, \vdash_{\mathbf{C}} \alpha_n$, and hence $\vdash_{\mathbf{C}} \alpha$, which is, according to theorem 2.7, a sufficient condition for concluding that $\Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$. □

3 Special Axiomatic Calculi

3.1 Definition. A calculus $\mathbf{C}$ is said to be *partial stable* if the following conditions are valid:

- each varying rule of $\mathbf{C}$ is unary, its domain is the collection of all formulas in $\mathbf{C}$, and each of its applications has exactly one varying object;
- for each application $\frac{\alpha'}{\alpha}$ of a varying rule in $\mathbf{C}$, if its varying object is not free in α', then $\alpha' \Vdash^{\emptyset}_{\mathbf{C}} \alpha$;
- for each application $\frac{\alpha_1,\ldots,\alpha_n}{\alpha}$ of a constant rule in $\mathbf{C}$, if $\alpha'_1, \ldots, \alpha'_n$ are respectively conclusions of applications of a varying rule over $\alpha_1, \ldots, \alpha_n$, using the same varying object, then $\alpha'_1, \ldots, \alpha'_n \Vdash^{\emptyset}_{\mathbf{C}} \alpha'$.

3.2 Theorem. If $\mathbf{C}$ is partial stable and $\Gamma \Vdash^{\emptyset}_{\mathbf{C}} \alpha$, then, for each application $\frac{\alpha}{\alpha'}$ of a varying rule in $\mathbf{C}$ such that its varying object is not free in Γ, $\Gamma \Vdash^{\emptyset}_{\mathbf{C}} \alpha'$.

Proof. It's similar to the proof of theorem 3.12. □

3.3 Theorem. If $\mathbf{C}$ is partial stable, then $\Gamma \vdash^{\emptyset}_{\mathbf{C}} \alpha$ iff $\Gamma \Vdash^{\emptyset}_{\mathbf{C}} \alpha$.

Proof. It's similar to the proof of theorem 3.13. □

3.4 Theorem. If $\mathbf{C}$ is partial stable, then "$\Vdash^{\emptyset}_{\mathbf{C}}$" has the following additional property:

- if
  - ∗ $\Gamma \Vdash^{\emptyset}_{\mathbf{C}} \alpha_1, \ldots, \Gamma \Vdash^{\emptyset}_{\mathbf{C}} \alpha_p$,
  - ∗ $\{\alpha_1, \ldots, \alpha_p\} \Vdash^{\mathbf{o_1},\ldots,\mathbf{o_n}}_{\mathbf{C}} \beta$,
  - ∗ for every $i \in \{1, \ldots, n\}$ and for every $j \in \{1, \ldots, p\}$, if $\mathbf{o_i}$ is free in α_j, then there exists $\Gamma' \subseteq \Gamma$ such that $\mathbf{o_i}$ is not free in Γ' and $\Gamma' \Vdash^{\emptyset}_{\mathbf{C}} \alpha_j$,

  then $\Gamma \Vdash^{\emptyset}_{\mathbf{C}} \beta$.

Proof. It is similar to the proof of theorem 3.14. □

3.5 Corollary. If $\mathbf{C}$ is partial stable, then the following additional properties are valid for the relation "$\overset{\emptyset}{\underset{\mathbf{C}}{\vdash}}$":

- $\Gamma \overset{\emptyset}{\underset{\mathbf{C}}{\vdash}} \alpha_1, \ldots, \Gamma \overset{\emptyset}{\underset{\mathbf{C}}{\vdash}} \alpha_p, \{\alpha_1, \ldots, \alpha_p\} \overset{\emptyset}{\underset{\mathbf{C}}{\vdash}} \beta$, then $\Gamma \overset{\emptyset}{\underset{\mathbf{C}}{\vdash}} \beta$;
- if $\begin{cases} * \ \Gamma \overset{\emptyset}{\underset{\mathbf{C}}{\vdash}} \alpha_1, \ldots, \Gamma \overset{\emptyset}{\underset{\mathbf{C}}{\vdash}} \alpha_p, \\ * \ \{\alpha_1, \ldots, \alpha_p\} \overset{\mathbf{o_1}, \ldots, \mathbf{o_n}}{\underset{\mathbf{C}}{\Vdash}} \beta, \\ * \ \text{for every } i \in \{1, \ldots, n\} \text{ and for every } j \in \{1, \ldots, p\}, \text{ if } \mathbf{o_i} \text{ is free in } \alpha_j, \text{ then there exists } \Gamma' \subseteq \Gamma \text{ such that } \mathbf{o_i} \text{ is not free in } \Gamma' \text{ and } \Gamma' \overset{\emptyset}{\underset{\mathbf{C}}{\vdash}} \alpha_j, \end{cases}$

  then $\Gamma \overset{\emptyset}{\underset{\mathbf{C}}{\vdash}} \beta$.

Proof. It suffices to use theorem 3.3, the eighth proposition of theorem 2.12 and theorem 3.4. □

3.6 Definition. A calculus $\mathbf{C}$ is said to be *partial strong* if the following clauses are satisfied:

- $\underset{\mathbf{C}}{\vdash} \alpha \to \alpha$;
- $\beta \overset{\emptyset}{\underset{\mathbf{C}}{\Vdash}} \alpha \to \beta$;
- $\alpha, \alpha \to \beta \overset{\emptyset}{\underset{\mathbf{C}}{\Vdash}} \beta$;
- for each application $\frac{\beta_1, \ldots, \beta_n}{\beta}$ of a constant rule in $\mathbf{C}$, $\{\alpha \to \beta_1, \ldots, \alpha \to \beta_n\} \overset{\emptyset}{\underset{\mathbf{C}}{\Vdash}} \alpha \to \beta$.

3.7 Theorem. The following propositions are equivalent:

- $\mathbf{C}$ is a partial strong calculus;
- for any Γ and α, whereon Γ is a collection of formulas in $\mathbf{C}$ and α is a formula in $\mathbf{C}$, $\Gamma \cup \{\alpha\} \overset{\emptyset}{\underset{\mathbf{C}}{\Vdash}} \beta$ iff $\Gamma \overset{\emptyset}{\underset{\mathbf{C}}{\Vdash}} \alpha \to \beta$.

Proof. It is similar to the proof of theorem 3.17. □

3.8 Scholium. If the first, second and fourth clauses of definition 3.6 are valid, then $\Gamma \cup \{\alpha\} \overset{\emptyset}{\underset{\mathbf{C}}{\Vdash}} \beta$ implies that $\Gamma \overset{\emptyset}{\underset{\mathbf{C}}{\Vdash}} \alpha \to \beta$.

3.9 Definition. A calculus is said to be *partial strong stable* if it is partial strong and partial stable.

3.10 Corollary.

- If $\mathbf{C}$ is a partial strong stable calculus, then $\Gamma \cup \{\alpha\} \overset{\emptyset}{\underset{\mathbf{C}}{\vdash}} \beta$ iff $\Gamma \overset{\emptyset}{\underset{\mathbf{C}}{\vdash}} \alpha \to \beta$.

Proof. It suffices to use theorems 3.3 and 3.7. □

3.11 Definition. A partial stable calculus **C** is said to be *stable* if it has the following additional property:

- for each application $\frac{\beta}{\alpha}$ of a varying rule in **C**, whereon **o** is its varying object, if α' and β' are respectively conclusions of applications of a varying rule in **C** over α and over β using a varying object distinct from **o**, then $\beta' \Vdash_{\mathbf{C}}^{\mathbf{o}} \alpha'$.

3.12 Theorem. If **C** is stable and $\Gamma \Vdash_{\mathbf{C}}^{\mathcal{V}} \alpha$, then, for each application $\frac{\alpha}{\alpha'}$ of a varying rule in **C** such that its varying object is not free in Γ, $\Gamma \Vdash_{\mathbf{C}}^{\mathcal{V}} \alpha'$.

Proof. Let $\frac{\alpha}{\alpha'}$ be an application of a varying rule in **C**, whose varying object, denoted by **o** from now on, is not free in Γ.
If α is an axiom of **C**, then $\vdash_{\mathbf{C}} \alpha$, so $\vdash_{\mathbf{C}} \alpha'$, therefore $\Gamma \Vdash_{\mathbf{C}}^{\mathcal{V}} \alpha'$.
If $\alpha \in \Gamma$, then **o** is not free in α, so, as **C** is stable, $\alpha \Vdash_{\mathbf{C}}^{\emptyset} \alpha'$, therefore $\Gamma \Vdash_{\mathbf{C}}^{\mathcal{V}} \alpha'$.
If there is an application of a constant rule $\frac{\alpha_1,\ldots,\alpha_n}{\alpha}$ in **C** such that $\Gamma \Vdash_{\mathbf{C}}^{\mathcal{V}} \alpha_1, \ldots, \Gamma \Vdash_{\mathbf{C}}^{\mathcal{V}} \alpha_n$, we have, by induction hypothesis, that $\Gamma \Vdash_{\mathbf{C}}^{\mathcal{V}} \alpha'_1, \ldots, \Gamma \Vdash_{\mathbf{C}}^{\mathcal{V}} \alpha'_n$, whereon $\alpha'_1, \ldots, \alpha'_n$ are respectively consequences of $\alpha_1, \ldots, \alpha_n$ by the same rule in which α' is a consequence of α, using the same varying object **o**. As **C** is stable, it follows that $\alpha'_1, \ldots, \alpha'_n \Vdash_{\mathbf{C}}^{\emptyset} \alpha'$, therefore $\Gamma \Vdash_{\mathbf{C}}^{\mathcal{V}} \alpha'$.
Let us suppose now that there exists an application $\frac{\beta}{\alpha}$ of a varying rule in **C**, whose varying object is **o**', such that $\Gamma \Vdash_{\mathbf{C}}^{\mathcal{V}} \beta$. Consider β' as consequence of β by the same rule in which α' is consequence of α, using the same varying object **o**. By induction hypothesis, $\Gamma \vdash_{\mathbf{C}}^{\mathcal{V}} \beta'$. If $\mathbf{o}' \in \mathcal{V}$ and $\mathbf{o}' = \mathbf{o}$, then $\mathbf{o} \in \mathcal{V}$, hence, from the hypothesis $\Gamma \vdash_{\mathbf{C}}^{\mathcal{V}} \alpha$, we have that $\Gamma \vdash_{\mathbf{C}}^{\mathcal{V}} \alpha'$. If $\mathbf{o}' \in \mathcal{V}$ and $\mathbf{o}' \neq \mathbf{o}$, then, as **C** is stable, $\beta' \Vdash_{\mathbf{C}}^{\mathbf{o}'} \alpha'$, therefore $\Gamma \Vdash_{\mathbf{C}}^{\mathcal{V}} \alpha'$. If $\mathbf{o}' \notin \mathcal{V}$, then $\vdash_{\mathbf{C}} \alpha$, hence $\vdash_{\mathbf{C}} \alpha'$, therefore $\Gamma \Vdash_{\mathbf{C}}^{\mathcal{V}} \alpha'$. □

3.13 Theorem. If **C** is stable, then $\Gamma \vdash_{\mathbf{C}}^{\mathcal{V}} \alpha$ iff $\Gamma \Vdash_{\mathbf{C}}^{\mathcal{V}} \alpha$.

Proof. By theorem 2.14, we have that $\Gamma \Vdash_{\mathbf{C}}^{\mathcal{V}} \alpha$ implies $\Gamma \vdash_{\mathbf{C}}^{\mathcal{V}} \alpha$, so it remains to prove the converse.
Let us suppose that $\Gamma \vdash_{\mathbf{C}}^{\mathcal{V}} \alpha$.
Let $\mathcal{D}$ be a demonstration of α from Γ depending on $\mathcal{V}$, β be the first occurrence of a formula in $\mathcal{D}$ justified as a consequence of an application of a varying rule $\frac{\beta'}{\beta}$ such that its varying object does not belong to $\mathcal{V}$ and

some premise is relevant to β' in $\mathcal{D}$, and $\mathbf{o}$ be the varying object of this application.
If $\mathbf{o}$ is not free in β', then, as $\mathbf{C}$ is stable, we have that $\beta' \mathrel{\Vert\!\frac{\emptyset}{\mathbf{C}}} \beta$, hence, as the considered occurrence of β' precedes β' in $\mathcal{D}$, we have that $\Gamma \mathrel{|\!\frac{\mathcal{V}}{\mathbf{C}}} \beta'$, and therefore, by transitivity of "$\Vert\!\frac{\mathcal{V}}{\mathbf{C}}$", $\Gamma \mathrel{\Vert\!\frac{\mathcal{V}}{\mathbf{C}}} \beta$.
If $\mathbf{o}$ is free in β', then, as $\mathbf{o} \notin \mathcal{V}$, there exists $\Gamma' \subseteq \Gamma$ such that $\mathbf{o}$ is not free in Γ' and $\Gamma' \mathrel{\Vert\!\frac{\mathcal{V}}{\mathbf{C}}} \beta'$, hence, as $\mathbf{C}$ is stable and in accordance with theorem 3.12, $\Gamma' \mathrel{\Vert\!\frac{\mathcal{V}}{\mathbf{C}}} \beta$, therefore $\Gamma \mathrel{\Vert\!\frac{\mathcal{V}}{\mathbf{C}}} \beta$.
In any case, there is a demonstration $\mathcal{D}_\beta$ in $\mathbf{C}$ of β from Γ supported by $\mathcal{V}$. Replacing the considered occurrence of β in $\mathcal{D}$ by $\mathcal{D}_\beta$, we obtain, given $\mathcal{D}$, a demonstration in $\mathbf{C}$ of α from Γ, in which the number of applications of varying rules, whose varying objects do not belong to $\mathcal{V}$ and whose hypotheses have premises relevant to them in the new demonstration, has decreased one unit. Repeating the same process a finite number of times, we obtain a demonstration in $\mathbf{C}$ of α from Γ supported by $\mathcal{V}$, or rather $\Gamma \mathrel{\Vert\!\frac{\mathcal{V}}{\mathbf{C}}} \alpha$. □

3.14 Theorem. If $\mathbf{C}$ is stable, then "$\Vert\!\frac{\mathcal{V}}{\mathbf{C}}$" has the following additional property:

- if $\begin{cases} * \ \Gamma \mathrel{\Vert\!\frac{\mathcal{V}}{\mathbf{C}}} \alpha_1, \ldots, \Gamma \mathrel{\Vert\!\frac{\mathcal{V}}{\mathbf{C}}} \alpha_p, \\ * \ \{\alpha_1, \ldots, \alpha_p\} \mathrel{\Vert\!\frac{\mathbf{o_1}, \ldots, \mathbf{o_n}}{\mathbf{C}}} \beta, \\ * \ \text{for every } i \in \{1, \ldots, n\} \text{ and for every } j \in \{1, \ldots, p\}, \text{ if } \mathbf{o_i} \notin \mathcal{V} \\ \quad \text{and } \mathbf{o_i} \text{ is free in } \alpha_j, \text{ then there exists } \Gamma' \subseteq \Gamma \text{ such that } \mathbf{o_i} \text{ is not} \\ \quad \text{free in } \Gamma' \text{ and } \Gamma' \mathrel{\Vert\!\frac{\mathcal{V}}{\mathbf{C}}} \alpha_j, \end{cases}$

  then $\Gamma \mathrel{\Vert\!\frac{\mathcal{V}}{\mathbf{C}}} \beta$.

Proof. Let $\mathcal{D}_1, \ldots, \mathcal{D}_p$ be respectively demonstrations in $\mathbf{C}$ of $\alpha_1, \ldots, \alpha_p$ from Γ supported by $\mathcal{V}$, and let $\mathcal{E}$ be a demonstration in $\mathbf{C}$ of β from $\{\alpha_1, \ldots, \alpha_p\}$ supported by $\{\mathbf{o_1}, \ldots, \mathbf{o_n}\}$. Concatenating $\mathcal{D}_1, \ldots, \mathcal{D}_p, \mathcal{E}$, we obtain a demonstration $\mathcal{D}$ of β in $\mathbf{C}$ from Γ.
Let γ be the first occurrence of a formula in $\mathcal{D}$ justified as a consequence of an application $\frac{\gamma'}{\gamma}$ of a varying rule, such that its varying object does not belong to $\mathcal{V}$ and some element of Γ is relevant to γ' in $\mathcal{D}$. As $\mathcal{D}_1, \ldots, \mathcal{D}_p$ are demonstrations supported by $\mathcal{V}$, we have that the considered occurrence of γ' appears in $\mathcal{E}$, and hence, considering $\mathbf{o}$ the varying object of the application, we get $\mathbf{o} \in \{\mathbf{o}_1, \ldots, \mathbf{o}_n\}$.
Let ϑ and ζ be defined by

$$\vartheta = \{\alpha_j \mid j \in \{1, \ldots, p\} \text{ and } \mathbf{o} \text{ is free in } \alpha_j\},$$
$$\zeta = \{\alpha_j \mid j \in \{1, \ldots, p\} \text{ and } \mathbf{o} \text{ is not free in } \alpha_j\}.$$

It is easy to verify that there exists a finite Γ', such that $\Gamma' \subseteq \Gamma$, $\mathbf{o}$ is not free in Γ' and, for every $\delta \in \vartheta$, $\Gamma' \Vdash^{\mathcal{V}}_{\mathbf{C}} \delta$. Therefore, by the construction of ζ, $\Gamma' \cup \zeta \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_1, \ldots, \Gamma' \cup \zeta \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_p$, and $\mathbf{o}$ is not free in $\Gamma' \cup \zeta$.
As the considered occurrence of γ' precedes γ in $\mathcal{D}$, we have that $\{\alpha_1, \ldots, \alpha_p\} \Vdash^{\mathcal{V}}_{\mathbf{C}} \gamma'$, and hence, by transitivity of "$\Vdash^{\mathcal{V}}_{\mathbf{C}}$", we get $\Gamma' \cup \zeta \Vdash^{\mathcal{V}}_{\mathbf{C}} \gamma'$, and therefore, by theorem 3.12, $\Gamma' \cup \zeta \Vdash^{\mathcal{V}}_{\mathbf{C}} \gamma$.
For every $\delta \in \Gamma' \cup \zeta$, we have that $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \delta$, and hence, once again due to transitivity of "$\Vdash^{\mathcal{V}}_{\mathbf{C}}$", $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \gamma$. Or rather, there exists a demonstration $\mathcal{D}_\gamma$ in $\mathbf{C}$ of γ from Γ supported by $\mathcal{V}$. Replacing the considered occurrence of γ in $\mathcal{D}$ by $\mathcal{D}_\gamma$, we have a new demonstration $\mathcal{D}'$ in $\mathbf{C}$ of β from Γ, in which the number of applications of varying rules, whose varying objects do not belong to $\mathcal{V}$ and each hypothesis has some premise relevant to it in $\mathcal{D}'$, has decreased one unit. Repeating the same process a finite number of times, we obtain a demonstration in $\mathbf{C}$ of β from Γ supported by $\mathcal{V}$, or rather, $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \beta$. □

3.15 Corollary. If $\mathbf{C}$ is stable, then the following additional properties are valid for the relation "$\vdash^{\mathcal{V}}_{\mathbf{C}}$":

- if $\Gamma \vdash^{\mathcal{V}_1}_{\mathbf{C}} \alpha_1, \ldots, \Gamma \vdash^{\mathcal{V}_p}_{\mathbf{C}} \alpha_p, \{\alpha_1, \ldots, \alpha_p\} \vdash^{\mathcal{W}}_{\mathbf{C}} \beta$, then $\Gamma \vdash^{\mathcal{V}_1 \cup \cdots \cup \mathcal{V}_p \cup \mathcal{W}}_{\mathbf{C}} \beta$;
- if
  - $*$ $\Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_1, \ldots, \Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_p$,
  - $*$ $\{\alpha_1, \ldots, \alpha_p\} \vdash^{\mathbf{o}_1, \ldots, \mathbf{o}_n}_{\mathbf{C}} \beta$,
  - $*$ for every $i \in \{1, \ldots, n\}$ and for every $j \in \{1, \ldots, p\}$, if $\mathbf{o}_i \notin \mathcal{V}$ and $\mathbf{o}_i$ is free in α_j, then exists $\Gamma' \subseteq \Gamma$ such that $\mathbf{o}_i$ is not free in Γ' and $\Gamma' \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha_j$,

  then $\Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \beta$.

Proof. It suffices to use theorem 3.13, the eight proposition of theorem 2.12 and theorem 3.14. □

3.16 Definition. A partial strong calculus $\mathbf{C}$ is said to be *strong* if it has the following additional property:

- for each application $\frac{\beta_1, \ldots, \beta_n}{\beta}$ of a varying rule of $\mathbf{C}$ whose collection of varying objects is $\mathcal{V}$, if no element of $\mathcal{V}$ is free in α, then $\{\alpha \to \beta_1, \ldots, \alpha \to \beta_n\} \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha \to \beta$.

3.17 Theorem. The following propositions are equivalent:

(i) $\mathbf{C}$ is a strong calculus;

(ii) for any Γ, α and $\mathcal{V}$, whereon Γ is a collection of formulas in $\mathbf{C}$, α is a formula in $\mathbf{C}$ and each $\mathbf{o} \in \mathcal{V}$ is not free in α,
$\Gamma \cup \{\alpha\} \Vdash^{\mathcal{V}}_{\mathbf{C}} \beta$ iff $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha \to \beta$.

(i) implies (ii). Let us suppose that $\mathbf{C}$ is a strong calculus and that each $\mathbf{o} \in \mathcal{V}$ is not free in α.
If $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha \rightarrow \beta$, then, due to clause (iii) of definition 3.6, $\Gamma \cup \{\alpha\} \Vdash^{\mathcal{V}}_{\mathbf{C}} \beta$.
Suppose now that $\Gamma \cup \{\alpha\} \Vdash^{\mathcal{V}}_{\mathbf{C}} \beta$.
If β is an axiom of $\mathbf{C}$, then $\vdash_{\mathbf{C}} \beta$, hence, according to clause (ii) of definition 3.6, $\Gamma \vdash_{\mathbf{C}} \alpha \rightarrow \beta$, therefore $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha \rightarrow \beta$.
If $\beta \in \Gamma$, then $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \beta$, hence, according to clause (ii) of definition 3.6, $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha \rightarrow \beta$.
If $\beta = \alpha$, then, according to clause (i) of definition 3.6, $\Gamma \vdash_{\mathbf{C}} \alpha \rightarrow \alpha$, therefore $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha \rightarrow \beta$.
If there is an application $\frac{\beta_1,\ldots,\beta_n}{\beta}$ of a rule of $\mathbf{C}$ such that $\Gamma \cup \{\alpha\} \Vdash^{\mathcal{V}}_{\mathbf{C}} \beta_1, \ldots, \Gamma \cup \{\alpha\} \Vdash^{\mathcal{V}}_{\mathbf{C}} \beta_n$, we have, by induction hypothesis, that

$$\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha \rightarrow \beta_1, \ldots, \Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha \rightarrow \beta_n.$$

If there is a varying object of this application that does not belong to $\mathcal{V}$, then, according to theorem 2.10, $\vdash_{\mathbf{C}} \beta$, hence, once again by clause (ii) of definition 3.6, $\vdash_{\mathbf{C}} \alpha \rightarrow \beta$, therefore $\Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha \rightarrow \beta$. If every varying object of this application belongs to $\mathcal{V}$, then, as $\mathbf{C}$ is strong, we conclude that $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha \rightarrow \beta$. □

(ii) implies (i). Let us suppose that for any Γ, α and $\mathcal{V}$, whereon Γ is a collection of formulas in $\mathbf{C}$, α is a formula in $\mathbf{C}$ and each $\mathbf{o} \in \mathcal{V}$ is not free in α, $\Gamma \cup \{\alpha\} \Vdash^{\mathcal{V}}_{\mathbf{C}} \beta$ iff $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha \rightarrow \beta$.
As $\alpha \vdash_{\mathbf{C}} \alpha$, we have that $\vdash_{\mathbf{C}} \alpha \rightarrow \alpha$.
As $\{\beta, \alpha\} \Vdash^{\emptyset}_{\mathbf{C}} \beta$, we get $\beta \Vdash^{\emptyset}_{\mathbf{C}} \alpha \rightarrow \beta$.
As $\alpha \rightarrow \beta \Vdash^{\emptyset}_{\mathbf{C}} \alpha \rightarrow \beta$, we have that $\{\alpha, \alpha \rightarrow \beta\} \Vdash^{\emptyset}_{\mathbf{C}} \beta$.
Finally, let $\frac{\beta_1,\ldots,\beta_n}{\beta}$ be an application of a rule of $\mathbf{C}$ whose collection of varying objects is $\mathcal{V}$, and α a formula in $\mathbf{C}$ where no element of $\mathcal{V}$ is free.
We have that

$$\{\alpha \rightarrow \beta_1, \ldots, \alpha \rightarrow \beta_n, \alpha\} \Vdash^{\emptyset}_{\mathbf{C}} \beta_1, \ldots, \{\alpha \rightarrow \beta_1, \ldots, \alpha \rightarrow \beta_n, \alpha\} \Vdash^{\emptyset}_{\mathbf{C}} \beta_n,$$

hence, as $\{\beta_1, \ldots, \beta_n\} \Vdash^{\mathcal{V}}_{\mathbf{C}} \beta$,

$$\{\alpha \rightarrow \beta_1, \ldots, \alpha \rightarrow \beta_n, \alpha\} \Vdash^{\mathcal{V}}_{\mathbf{C}} \beta,$$

therefore

$$\{\alpha \rightarrow \beta_1, \ldots, \alpha \rightarrow \beta_n\} \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha \rightarrow \beta.$$

□

3.18 Scholium. If the first, second and fourth clauses of definition 3.6, together with the only clause of definition 3.16, are valid, then $\Gamma \cup \{\alpha\} \Vdash^{\mathcal{V}}_{\mathbf{C}} \beta$ implies that $\Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha \to \beta$.

3.19 Definition. A calculus is said to be *strong stable* if it is strong and stable.

3.20 Theorem. If $\mathbf{C}$ is a strong stable calculus and each $\mathbf{o} \in \mathcal{V}$ is not free in α, then $\Gamma \cup \{\alpha\} \vdash^{\mathcal{V}}_{\mathbf{C}} \beta$ iff $\Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha \to \beta$.

Proof. It suffices to use theorems 3.17 and 3.13. □

3.21 Definition. We note by $\mathbf{C}[\Gamma]$ the calculus obtained from $\mathbf{C}$ with the addition of Γ as a postulate. If Γ is a singleton of the form $\{\alpha\}$, then we also note $\mathbf{C}[\Gamma]$ by $\mathbf{C}[\alpha]$.

3.22 Theorem. The following assertions are valid for $\mathbf{C}[\Gamma]$:

- $\Gamma' \cup \Gamma \vdash_{\mathbf{C}} \alpha$ iff $\Gamma' \vdash_{\mathbf{C}[\Gamma]} \alpha$;
- $\Gamma' \cup \Gamma \Vdash^{\emptyset}_{\mathbf{C}} \alpha$ iff $\Gamma' \Vdash^{\emptyset}_{\mathbf{C}[\Gamma]} \alpha$;
- if $\Gamma' \cup \Gamma \vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$, then $\Gamma' \vdash^{\mathcal{V}}_{\mathbf{C}[\Gamma]} \alpha$;
- if $\Gamma' \vdash^{\mathcal{V}}_{\mathbf{C}[\Gamma]} \alpha$, then there exists $\mathcal{W} \supseteq \mathcal{V}$ such that $\Gamma' \cup \Gamma \vdash^{\mathcal{W}}_{\mathbf{C}} \alpha$;
- if $\Gamma' \cup \Gamma \Vdash^{\mathcal{V}}_{\mathbf{C}} \alpha$, then $\Gamma' \Vdash^{\mathcal{V}}_{\mathbf{C}[\Gamma]} \alpha$;
- if $\Gamma' \vdash^{\mathcal{V}}_{\mathbf{C}[\Gamma]} \alpha$, then there exists $\mathcal{W} \supseteq \mathcal{V}$ such that $\Gamma' \cup \Gamma \vdash^{\mathcal{W}}_{\mathbf{C}} \alpha$;
- if $\mathbf{C}$ is partial stable, then $\mathbf{C}[\Gamma]$ is partial stable;
- if $\mathbf{C}$ is partial strong, then $\mathbf{C}[\Gamma]$ is partial strong;
- if $\mathbf{C}$ is partial strong stable, then $\mathbf{C}[\Gamma]$ is partial strong stable;
- if $\mathbf{C}$ is stable, then $\mathbf{C}[\Gamma]$ is stable;
- if $\mathbf{C}$ is strong, then $\mathbf{C}[\Gamma]$ is strong;
- if $\mathbf{C}$ is strong stable, then $\mathbf{C}[\Gamma]$ is strong stable.

3.23 Corollary. $\mathbf{C}[\Gamma]$ has the following properties with respect to introduction of implication:

- if $\mathbf{C}$ is partial strong, then $\Gamma' \cup \{\alpha\} \Vdash^{\emptyset}_{\mathbf{C}[\Gamma]} \beta$ iff $\Gamma' \Vdash^{\emptyset}_{\mathbf{C}[\Gamma]} \alpha \to \beta$;
- if $\mathbf{C}$ is partial strong stable, then $\Gamma' \cup \{\alpha\} \vdash^{\emptyset}_{\mathbf{C}[\Gamma]} \beta$ iff $\Gamma' \vdash^{\emptyset}_{\mathbf{C}[\Gamma]} \alpha \to \beta$;
- if $\mathbf{C}$ is strong and each $\mathbf{o} \in \mathcal{V}$ is not free in α, then $\Gamma' \cup \{\alpha\} \Vdash^{\mathcal{V}}_{\mathbf{C}[\Gamma]} \beta$ iff $\Gamma' \Vdash^{\mathcal{V}}_{\mathbf{C}[\Gamma]} \alpha \to \beta$;
- if $\mathbf{C}$ is strong stable and each $\mathbf{o} \in \mathcal{V}$ is not free in α, then $\Gamma' \cup \{\alpha\} \vdash^{\mathcal{V}}_{\mathbf{C}[\Gamma]} \beta$ iff $\Gamma' \vdash^{\mathcal{V}}_{\mathbf{C}[\Gamma]} \alpha \to \beta$.

4 Conclusions

We have found optimized formulations for the deduction theorem for a broad class of open axiomatic calculi, which overcome all the problems that we pointed out in the beginning, within every spectrum of possible restrictions in their deductive functioning — from the partial stable and partial strong to the strong stable calculi.

The weakest formulation of the deduction theorem belongs among the partial strong stable calculi. An example of a calculus of this type can be seen in [2], p. 133, which is a translation to a first order language of the *Logic of Skeptical Deduction*, defined in the same work, in Chapter 5. This calculus was essential for the proof of completeness of an axiomatic calculus with respect to the semantics of this logic.

The strongest formulation of the same theorem belongs among the strong stable calculi, which constitute the great majority of open axiomatic calculi, concerning the material implication, found in the literature.

Our initial motivation was the search for a conceptual basis for an abstract proof of completeness regarding generic calculi, which was done in [2], pp. 72–88. A concise exposition of this proof will be the subject of a future paper.

References

[1] John Bell and Moshé Machover. *A Course in Mathematical Logic.* North-Holland, 1977.

[2] Arthur Buchsbaum. *Lógicas da Inconsistência e da Incompletude: Semântica e Axiomática.* PhD thesis, Pontifícia Universidade Católica do Rio de Janeiro, 1995.

[3] Arthur Buchsbaum and Tarcisio Pequeno. A logic for careful non-monotonic reasoning. 1996.

[4] H.-D. Ebbinghaus, J. Flum, and W. Thomas. *Mathematical Logic.* Springer-Verlag, 1994.

[5] Herbert B. Enderton. *A Mathematical Introduction to Logic.* Academic Press, 1972.

[6] Stephen Cole Kleene. *Introduction to Metamathematics.* Wolters-Noordhoff, North Holland and American Elsevier, 1974.

[7] Elliott Mendelson. *Introduction to Mathematical Logic*. D. Van Nostrand, 1979.

[8] Dag Prawitz. *Natural Deduction: A Proof-Theoretical Study*. Almqvist & Wiksell, 1965.

[9] Joseph R. Shoenfield. *Mathematical Logic*. Addison-Wesley, 1967.

[10] Dirk van Dalen. *Logic and Structure*. D. Reidel, 1989.