A Logic for Ambiguous Description

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Abstract

A logic formalizing ambiguity, which appears both in natural language and in mathematical discourse, is presented, through a sequent calculus and a semantics, together with some elementary results.

1 Introduction

There are almost an infinite number of situations in mathematics, logic and everyday speech in which we have more than one object satisfying a given property, and we would like to use a name to denote an arbitrary object of this class.

So, in mathematics, for example, we denote a primitive of the function defined by $f(x) = 2x$ by $\int 2x\,dx$, although we know that there exists more than one primitive for this function.

In syntax of formal logic, we usually define the expression $\exists ! x\, P$ by

$$\exists x\, P \wedge \forall x \forall y \left(P \wedge P(x|y) \rightarrow x = y\right),$$

whereon y is the first variable distinct of x which does not occur in P. It would be more natural to consider the expression $\exists ! x\, P$ as an ambiguous reference for any formula of that form, whereon it is only requested that y is distinct from x and it does not occur in P, dropping out the restriction about the alphabetical position of y.

In everyday speech any noun preceded by an indefinite article is an ambiguous reference for any object of the correspondent collection. For example, the expression "a flower" is an ambiguous reference for any specific

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flower. So, the expression "a flower is beautiful" means, in a possible sense, that any flower is beautiful.

Besides ambiguous descriptions, there is a kind of assertions saying that a given object corresponds to some description.

In mathematics, by an abusive usage of the equality sign, we say that the function defined by $g(x) = x^2 + 3$ is a primitive of $f(x) = 2x$ by writing "$\int 2x\,dx = x^2 + 3$".

In everyday speech, when we want to say that a rose is referenced by the description "a flower", we utter "a rose is a flower".

So, we have isolated two key ideas concerning to ambiguous reference: *description* and *comprising*. The symbols used in this text for description and comprising are respectively "Υ" and "$\models$".

Roughly speaking, according to our notation, we have:

- "$\int 2x\,dx$" is a shorthand for
$$\Upsilon g\,\big(g \text{ is a primitive of the function } f(x) = 2x\big);$$
- we can say that the function $g(x) = x^2 + 3$ is a primitive of $f(x) = 2x$ by writing "$\int 2xdx \models g$" or "$\int 2xdx \models (x^2+3)dx$"; the reader should note the use of the sign "$\models$"instead of the equality sign, as it is usually done, in a wrong way;
- we can also say "a rose is a flower" by the expression
$$\Upsilon x\,(x \text{ is a flower}) \models \Upsilon x\,(x \text{ is a } \textit{rose}).$$

A logic for dealing with these two ideas, enriching classical logic, is defined here, from now on named *Logic of Ambiguous Reference*, shortly LAR. We have defined a semantics and a sequent calculus for LAR, fitting to some basic intuitions. We also present some basic results concerning semantics and proof theory.

According to our intuition, such logic should take into account the following perspectives:

- a description "$\Upsilon x\,P$" should comprise, under reasonable restrictions, every term satisfying P, and only these terms;
- there should be a replacement rule for comprising, or, in a more formal way, $\dfrac{\Gamma \vdash t \models t'}{\Gamma, P(x\|t) \vdash P(x\|t')}$, under reasonable restrictions, should be a rule of LAR;
- LAR should work as close as possible to classical logic, since the above conditions be respected;
- LAR should be a conservative extension of classical logic.

Another remarkable quality of LAR is that it doesn't adopt *equality* as a primitive concept. Equality is instead a concept derived from *comprising*. When we have two descriptions, each comprising the other, we say that they are *equicomprising* or *equivalent*, and we will use the sign "=" for formalizing this situation.

2 A Language for LAR

In this section some syntactical details related to the meaningful expressions of LAR are provided. They are used everywhere in this paper, from results and definitions related to semantics, to the rules and theorems related to the sequent calculus of LAR.

2.1 Definition. A language for LAR has all the signs of a standard first order language, without equality, having "$\rightarrow$", "$\wedge$", "$\vee$" and "$\neg$" as connectives and "$\forall$" and "$\exists$" as quantifiers, plus the sign "Υ" as a *qualifier*, the adopted sign for *ambiguous description*, and the sign "$)\!\!=$"as a *special binary predicate sign*, the adopted sign for *comprising*.

2.2 Definition. Terms and formulas in LAR are all the terms and formulas in a standard first order language[1], plus the following ones:

- if x is a variable and P is a formula, then $\Upsilon x\, P$ is a term in LAR, also called a *description*;
- if t and t' are terms in LAR, then $t \,)\!\!= t'$ is a formula in LAR.

Terms and formulas in LAR are also called *designators in* LAR.

Unless stated otherwise, for some syntactic variables, with or without primes and subscripts, there are established special usages: c is a constant; x,y,z are variables; f,g,h are function signs; p,q,r are predicate signs; t,u,v are terms in LAR; P,Q,R,S,T are formulas in LAR, D,E are designators in LAR, Γ, Φ are collections of formulas in LAR, and L is a language for LAR.

2.3 Definition. An occurrence of a variable in D is said *bound in* D if it occurs inside a subdesignator of D of one of the forms $\forall x\, P$, $\exists x\, P$ or $\Upsilon x\, P$. An occurrence of a variable in D is said *free* in D if it is not bound in D. A variable is said *free in* D if it has at least a free occurrence in D.

2.4 Definition. An occurrence of a designator D in a designator D' is said *real in* D' if it doesn't succeed "$\forall$", "$\exists$"or "Υ"[2] in D'.

[1] See [8], for example.
[2] It can happen if D is a variable.

2.5 Definition. A variable x is said *to accept* a term t in a designator D if D has no subdesignator of one of the forms $\forall y\, P$, $\exists y\, P$ or $\Upsilon y\, P$, in which x has an occurrence in P free in D such that y is free in t.

2.6 Definition. A designator D is said *to be in the scope of a variable x in a designator D'* if there is a subdesignator in D' of one of the forms $\forall x\, P$, $\exists x\, P$ or $\Upsilon x\, P$, such that there is a real occurrence of D in P; otherwise D is said *to be out of the scope of x in D'*.

2.7 Definition.

- $D(x|t)$ denotes the designator obtained from D by substituting t for each free occurrence of x, and replacing consistently bound occurrences of variables in D for other ones don't occurring in D when it is necessary;[3]
- $E(D\|D')$ denotes the designator obtained from E by replacing all real occurrences of D by D'.

2.8 Definition. A designator in LAR is said *pure* if it has no occurrence of " Υ " outside of the scope of " $\models$ ".

2.9 Definition. An occurrence of a designator in a designator in LAR is said a *top occurrence* if it's real and it's out of the scopes of " Υ "and " $\models$ ".

2.10 Definition. A variable x is said *top in* a designator D if all free occurrences of x in D are top occurrences.

2.11 Definition. A formula having no top occurrence of some formula of the form " $u \models v$ " is said a *basic formula.*

3 A Semantics for LAR

3.1 Definition. A *simple structure for L* is a pair $\langle \Delta, \sigma \rangle$,whereon Δ is a non empty set, called the *universe of* the structure, and σ is a function, called the *sign assignment of* the structure, whose domain is the collection of constants, function signs and predicate signs in L, obeying the following conditions:

- $\sigma(c)$ is an element of Δ;
- if n is the arity of f, $\sigma(f)$ is a function from Δ^n to Δ;
- if n is the arity of p, $\sigma(p)$ is a subset of Δ^n.

A LAR-*structure for L* is a simple structure for L.

[3] It avoids that free occurrences of variables in t become bound in $D(x|t)$.

3.2 Definition. Let $\mathfrak{A} = \langle \Delta, \sigma \rangle$ be a simple structure for L. A Δ-*assignment* is a function from the collection of variables in L to Δ. A simple interpretation for L is a pair $\langle \mathfrak{A}, s \rangle$, whereon s is a Δ-assignment, also called the *variable assignment of* the interpretation. A LAR-*interpretation for* L is a simple interpretation for L.

3.3 Definition. Let s be a Δ-assignment and d be an element of Δ. $s(x|d)$ denotes the Δ-assignment defined from s by

$$s(x|d)(y) = \begin{cases} s(y), & \text{if } y \text{ is distinct from } x; \\ d, & \text{if } y \text{ is } x. \end{cases}$$

If $I = \langle \Delta, \sigma, s \rangle$ is a simple interpretation, then $I(x|d)$ denotes the interpretation $\langle \Delta, \sigma, s(x|d) \rangle$. Consider also defined $s(x_1, \ldots, x_n|d_1, \ldots, d_n)$ and $I(x_1, \ldots, x_n|d_1, \ldots, d_n)$.

Next a semantics for LAR is provided. For expressing possible ambiguity, each term is associated, by the function I_D defined below, with a set of elements of the universe of discourse, in the same sense by which, for example, in a natural language like English, the expression "an orange" is associated with the set of all oranges, although, of course, "an orange" does not mean the set of all oranges, but it is an ambiguous representation for an arbitrary orange of this set. This set can be empty; in this case the term is said to be *vacuous*, there is, it's a name for nothing. For example, "$\Upsilon x\,(x \neq x)$" is a *vacuous term*, according to the usual meaning ascribed to the sign "$\neq$", whereas "$\Upsilon x\,(x \in \mathbb{N} \wedge x > 2)$" is instead an *ambiguous term*, and "$\Upsilon x\,(x \in \mathbb{N} \wedge x \text{ is even} \wedge x \text{ is prime})$" is a *univocal term*.

Let P be a basic formula such that $x_1, \ldots, x_n$ are distinct variables top in P, and consider $P(x_1, \ldots, x_n|t_1, \ldots, t_n)$ the formula obtained from P by simultaneous replacement of $x_1, \ldots, x_n$ by $t_1, \ldots, t_n$.

The true values of LAR are *victory* (or *true*) and *defeat* (or *false*), represented here by 1 and 0.

The function I_S, defined below, is a LAR-valuation, that is, it is the function which assigns a true value for each formula, whereas the function I_N, which also assigns a true value for each formula, is an auxiliary one, used for defining I_S in a simultaneous recursive way.

We say that $d_1, \ldots, d_n$ *satisfy* $P(x_1, \ldots, x_n)$ (according to a given simple interpretation I) if $I(x_1, \ldots, x_n|d_1, \ldots, d_n)_S(P) = 1$.

If, according to I_D, each of the terms $t_1, \ldots, t_n$ denotes at least one object, then I_S assigns *victory* to $P(x_1, \ldots, x_n|t_1, \ldots, t_n)$ if, and only if, for each $d_1, \ldots, d_n$, such that $d_1, \ldots, d_n$ are respectively elements of the universe of discourse denoted (ambiguously) by $t_1, \ldots, t_n$, $d_1, \ldots, d_n$ satisfy P.

If some of these terms denotes no object, according to I_D, and P is an atomic formula, then $P(x_1,\ldots,x_n|t_1,\ldots,t_n)$ is evaluated as a (*vacuous*) *victory* according to I_S.

I_N presents, in a sense, a complementary behavior. If each of the terms $t_1,\ldots,t_n$ denotes at least one object, then I_N assigns *victory* to $P(x_1,\ldots,x_n|t_1,\ldots,t_n)$ if, and only if, for all objects $d_1,\ldots,d_n$ denoted by $t_1,\ldots,t_n$, $d_1,\ldots,d_n$ don't satisfy P. If some of these terms denotes no object, and P is an atomic formula, then $P(x_1,\ldots,x_n|t_1,\ldots,t_n)$ is evaluated as a (*vacuous*) *victory* according to I_N.

This kind of semantics was inspired by our previous work about *paraconsistent* and/or *paracomplete* logics [1, 9] and, recently, by some ideas about game based semantics [2, 7]. The letter "S" in "I_S" comes from the word "subject", whereas the letter "N" in "I_N" comes from the word "nature". The basic idea is relative to an imaginary game between the *subject*, who wants to prove that a given formula is true, and the *nature*, who wants to prove that the negation of this formula is true.

3.4 Definition. Let $I = \langle \Delta, \sigma, s\rangle$ be a LAR-interpretation for L. The following clauses specify the functions I_D, I_S and I_N, whereon I_D is said the LAR-*denotation for L defined by I*, and I_S is said the LAR-*valuation for L defined by I*:

- I_D is a function from the collection of terms in L to $\mathcal{P}(\Delta)$;
- I_S, I_N are functions from L to {0,1};
- $I_D(c) = \{\sigma(c)\}$;
- $I_D(x) = \{s(x)\}$;
- $I_D(f(t_1,\ldots,t_n)) = \{\sigma(f)(d_1,\ldots,d_n)|d_1 \in I_D(t_1),\ldots,d_n \in I_D(t_n)\}$;
- $I_D(\Upsilon x\, P) = \{d \in \Delta \mid I(x|d)_S(P) = 1\}$;
- $I_S(p(t_1,\ldots,t_n)) = 1$ iff for each $d_1 \in I_D(t_1),\ldots,$ for each $d_n \in I_D(t_n)$, $\langle d_1,\ldots,d_n\rangle \in \sigma(p)$;
- $I_N(p(t_1,\ldots,t_n)) = 1$ iff for each $d_1 \in I_D(t_1),\ldots,$ for each $d_n \in I_D(t_n)$, $\langle d_1,\ldots,d_n\rangle \notin \sigma(p)$;
- $I_S(t \models t') = 1$ iff $I_N(t \models t') = 0$ iff $I_D(t) \supseteq I_D(t')$;
- $I_S(\neg P) = I_N(P)$;
- $I_N(\neg P) = I_S(P)$;
- $I_S(P \to Q) = \max\{I_N(P), I_S(Q)\}$;
- $I_N(P \to Q) = \min\{I_S(P), I_N(P)\}$;
- $I_S(P \wedge Q) = \min\{I_S(P), I_S(Q)\}$;
- $I_N(P \wedge Q) = \max\{I_N(P), I_N(Q)\}$;
- $I_S(P \vee Q) = \max\{I_S(P), I_S(Q)\}$;
- $I_N(P \vee Q) = \min\{I_N(P), I_N(Q)\}$;

- $I_S(\forall x\, P) = \min\{I(x|d)_S(P) \mid d \in \Delta\}$;
- $I_N(\forall x\, P) = \max\{I(x|d)_N(P) \mid d \in \Delta\}$;
- $I_S(\exists x\, P) = \max\{I(x|d)_S(P) \mid d \in \Delta\}$;
- $I_N(\exists x\, P) = \min\{I(x|d)_N(P) \mid d \in \Delta\}$.

This semantics reflects a *non alethic logic* (a logic that is both *paraconsistent* and *paracomplete*), there is, a logic in which both P and $\neg P$ can be true (both the subject and the nature can win; it is shared by all *paraconsistent logics*), or in which both P and $\neg P$ can be false (both the subject and the nature can lose; it is shared by all *paracomplete logics*). Classical references for this kind of logics can be found in [3, 5, 4]. Besides being non alethic, LAR is also a *non reflexive logic*, that is, it is a logic in which "$P \to P$" can be false.

3.5 Definition. A term t is said *vacuous with respect to* a simple interpretation I if $I_D(t)$ is the empty set, *existential* if $I_D(t)$ is non empty, *univocal* if $I_D(t)$ is a singleton, and *ambiguous* if $I_D(t)$ has at least two members.

3.6 Example. Consider $p(x)$ a basic atomic formula in which x is top, and I a simple interpretation.

- If t is vacuous with respect to I, then both $p(x|t)$ and $\neg p(x|t)$ are true according to I_S, so confirming the *paraconsistency* of LAR. For example, both the formula "$\Upsilon x\,(x \neq x)$ is even" and its negation are true.
- If t is ambiguous according to I such that there are d_1 and d_2 belonging to $I_D(t)$ for which $I(x|d_1)_S(p(x)) = 1$ and $I(x|d_2)_S(p(x)) = 0$, then both "$p(x|t)$" and "$\neg p(x|t)$" are false according to I_S, so confirming the *paracompleteness* of LAR. It also happens in this case that "$p(x|t) \to p(x|t)$" is false, so confirming the *non reflexiveness* of LAR. For example, the formula "$\Upsilon x\,(x = 1 \vee x = 2)$ is even" is false together with its negation, and

  $$\Upsilon x\,(x = 1 \vee x = 2) \text{ is even} \to \Upsilon x\,(x = 1 \vee x = 2) \text{ is even}$$

  is false too.

3.7 Definition. LAR-*satisfiability*, LAR-*validity* and LAR-*consequence* are defined in the same way it is done in classical logic. For example, LAR-*consequence* is defined by the following clause:

- P is a LAR-*consequence of* Γ if every LAR-valuation satisfying Γ also satisfies P; we denote it here by $\Gamma \models P$.

Next a basic semantic result concerning instantiation is provided.

3.8 Theorem. Let I be a simple interpretation for L.

(i) If t is a pure term, then $\begin{cases} * \ I_D(u(x|t)) = I(x|t^I)_D(u); \\ * \ I_S(P(x|t)) = I(x|t^I)_S(P); \\ * \ I_N(P(x|t)) = I(x|t^I)_N(P), \text{ whereon } t^I \text{ is} \\ \quad \text{the unique element of the singleton } I_D(t). \end{cases}$

(ii) If x is top in u, then $I_D(u(x|t)) \supseteq \bigcup_{d \in I_D(t)} I(x|d)_D(u)$.

(iii) If x is top in P, then $\begin{cases} I_S(P(x|t)) \leq \min\{I(x|d)_S(P) \mid d \in I_D(t)\}; \\ I_N(P(x|t)) \leq \min\{I(x|d)_N(P) \mid d \in I_D(t)\}. \end{cases}$

(iv) If $\begin{cases} x \text{ is top in } u, \\ x \text{ has only one free occurrence in } u, \end{cases}$

then $I_D(u(x|t)) = \bigcup_{d \in I_D(t)} I(x|d)_D(u)$.

(v) If $\begin{cases} x \text{ is top in } P, \\ x \text{ has only one free occurrence in } P, \\ I_D(t) \neq \emptyset, \end{cases}$

then $\begin{cases} I_S(P(x|t)) = \min\{I(x|d)_S(P) \mid d \in I_D(t)\}; \\ I_N(P(x|t)) = \min\{I(x|d)_N(P) \mid d \in I_D(t)\}. \end{cases}$

(vi) If $\begin{cases} x \text{ has a top occurrence in } u, \\ I_D(t) = \emptyset, \end{cases}$ then $I_D(u(x|t)) = \emptyset$.

(vii) If $\begin{cases} * \ P \text{ is a basic atomic formula or a negation} \\ \quad \text{of a basic atomic formula,} \\ * \ P \text{ has at least one top occurrence of } x, \\ * \ I_D(t) = \emptyset, \end{cases}$

then $I_S(P(x|t)) = I_N(P(x|t)) = 1$.

4 A Sequent Calculus for LAR

In this section LAR is characterized as a sequent calculus. Some basic syntactic results concerned with this sequent calculus are also provided.

4.1 Definition. We define when variables are free in a designator in an analogous way by which it is done in a standard first order language, taking in account that "Υ" is a variable binding term operator (or a qualifier).

4.2 Definition. From now on, together with the known defined signs "$\leftrightarrow$" and "$\neq$", we also adopt the following ones (consider x and y the first two variables which are not free in t):

- $t = t' \rightleftharpoons t \models t' \wedge t' \models t$;
- $\mathrm{vac}(t) \rightleftharpoons \neg\exists x \left(t \models x\right)$; "$\mathrm{vac}(t)$" is read "$t$ is *vacuous*";
- $\mathrm{ex}(t) \rightleftharpoons \exists x \left(t \models x\right)$; "$\mathrm{ex}(t)$" is read "$t$ is *existential*";
- $\mathrm{un}(t) \rightleftharpoons \forall x \forall y \left(t \models x \wedge t \models y \rightarrow x = y\right)$; "$\mathrm{un}(t)$" is read "$t$ is *univocal*";
- $\mathrm{amb}(t) \rightleftharpoons \exists x \exists y \left(t \models x \wedge t \models y \rightarrow x \neq y\right)$;
  "$\mathrm{amb}(t)$" is read "t is *ambiguous*".

Below we give the sequent rules of LAR, which characterize syntactically this logic.

4.3 Definition (Structural Rules).

- **Antecedent Rule**: If $\Gamma \subseteqq \Gamma'$, then $\dfrac{\Gamma \vdash P}{\Gamma' \vdash P}$.
- **Assumption Rule**: If $P \in \Gamma$, then $\Gamma \vdash P$.
- **Chain Rule**: $\dfrac{\Gamma \vdash P \qquad \Gamma, P \vdash Q}{\Gamma \vdash Q}$.

4.4 Definition (Connective Rules).

- **Modus Ponens**: If P is a pure formula, then $P, P \rightarrow Q \vdash Q$.
- **Deduction Rule**: If P is a pure formula, then $\dfrac{\Gamma, P \vdash Q}{\Gamma \vdash P \rightarrow Q}$.
- **∧-Elimination Rule**: $\begin{cases} P \wedge Q \vdash P; \\ P \wedge Q \vdash Q. \end{cases}$
- **∧-Introduction Rule**: $P, Q \vdash P \wedge Q$.
- **Proof by Cases Rule**: $\dfrac{\Gamma \vdash P \vee Q \qquad \Gamma, P \vdash R \qquad \Gamma, Q \vdash R}{\Gamma \vdash R}$.
- **∨-Introduction Rule**: $\begin{cases} P \vdash P \vee Q; \\ Q \vdash P \vee Q. \end{cases}$
- **Non Contradiction Rule**:
  ∗ If P and Q are pure formulas, then $\dfrac{\Gamma, P \vdash Q \qquad \Gamma, P \vdash \neg Q}{\Gamma \vdash \neg P}$.
- **Double Negation Rule**: $\begin{cases} \neg\neg P \vdash P; \\ P \vdash \neg\neg P. \end{cases}$

- **Material Implication Rule**: $\begin{cases} P \to Q \vdash \neg P \vee Q; \\ \neg P \vee Q \vdash P \to Q; \\ \neg(P \to Q) \vdash P \wedge \neg Q; \\ P \wedge \neg Q \vdash \neg(P \to Q). \end{cases}$
- **De Morgan Rule**: $\begin{cases} \neg(P \vee Q) \vdash \neg P \wedge \neg Q; \\ \neg P \wedge \neg Q \vdash \neg(P \vee Q); \\ \neg(P \wedge Q) \vdash \neg P \vee \neg Q; \\ \neg P \vee \neg Q \vdash \neg(P \wedge Q). \end{cases}$

4.5 Definition (Quantifier Rules).

- **∀-Elimination Rule**: If t is a pure term, then $\forall x\, P \vdash P(x|t)$.
- **Generalization Rule**: if x is not free in Γ, then $\dfrac{\Gamma \vdash P}{\Gamma \vdash \forall x\, P}$.
- **Witness Rule**: if y is not free in $\Gamma \cup \{\exists x\, P, Q\}$, then $\dfrac{\Gamma, P(x|y) \vdash Q}{\Gamma, \exists x\, P \vdash Q}$.
- **∃-Introduction Rule**: if t is a pure term, then $P(x|t) \vdash \exists x\, P$.
- **Alternation Rule**: $\begin{cases} \neg \exists x\, P \vdash \forall x\, \neg P; \\ \forall x\, \neg P \vdash \neg \exists x\, P; \\ \neg \forall x\, P \vdash \exists x\, \neg P; \\ \exists x\, \neg P \vdash \neg \forall x\, P. \end{cases}$

4.6 Definition (Comprising Rules).

- **Transitivity Rule**: $t \models u, u \models v \vdash t \models v$.
- **Extension Rule**:
  * If x is not free in t, t', then $\forall x \big(t \models x \to t' \models x\big) \vdash t' \models t$.
- **Globalization Rule**:
  * If $\begin{cases} x \text{ is not free in } t, \\ x \text{ is top in } P, \\ x \text{ has only one free occurrence in } P, \end{cases}$

    then $\mathrm{ex}(t), \forall x \big(t \models x \to P\big) \vdash P(x|t)$.
- **Replacement Rule**:
  * $t_1 \models u_1, \ldots, t_n \models u_n \vdash f(t_1, \ldots, t_n) \models f(u_1, \ldots, u_n)$;
  * $t_1 \models u_1, \ldots, t_n \models u_n, p(t_1, \ldots, t_n) \vdash p(u_1, \ldots, u_n)$;
  * $t_1 \models u_1, \ldots, t_n \models u_n, \neg p(t_1, \ldots, t_n) \vdash \neg p(u_1, \ldots, u_n)$.
- **Unity Rule**: If t,t' are pure terms, then $t \models t' \vdash t' \models t$.

- **Vacuity Rule**:
  - ∗ If $\begin{cases} P \text{ is a basic atomic formula or a negation of a basic atomic formula,} \\ P \text{ has at least one top occurrence of } x, \end{cases}$

    then $\mathrm{vac}(t) \vdash P(x|t)$.
- **Function Rule**:
  - ∗ If $\begin{cases} u \text{ is a pure term,} \\ x_1, \ldots, x_n \text{ are not free in } u, t_1, \ldots, t_n, \end{cases}$ then

$$\vdash f(t_1, \ldots, t_n) \mathrel{)\!\!=} u$$
$$\leftrightarrow$$
$$\exists x_1 \ldots \exists x_n \left(t_1 \mathrel{)\!\!=} x_1 \land \ldots \land t_n \mathrel{)\!\!=} x_n \land u = f(x_1, \ldots, x_n)\right).$$

- **Description Rule**: If t is a pure term, then $\begin{cases} \Upsilon x\, P \mathrel{)\!\!=} t \vdash P(x|t); \\ P(x|t) \vdash \Upsilon x\, P \mathrel{)\!\!=} t. \end{cases}$

Next some basic results about the sequent calculus for LAR are provided.

4.7 Theorem. Replacing in terms and formulas, with no occurrence of "Υ", the sign "$\mathrel{)\!\!=}$"for the sign "=", they behave in LAR as in classical equational logic.

Next two more kinds of implication are defined. The first one has modus ponens property and a corresponding deduction theorem, whereas the second one both modus ponens and modus tollens properties. For each one of these implications, it is also defined a corresponding equivalence.

4.8 Definition.
- $P \rightarrowtriangle Q \rightleftharpoons \Upsilon x\, Q \mathrel{)\!\!=} \Upsilon x\, P$,
  whereon x is the first variable not free in $\{P, Q\}$;
- $P \leftrightarrowtriangle Q \rightleftharpoons (P \rightarrowtriangle Q) \land (Q \rightarrowtriangle P)$;
- $P \Rightarrow Q \rightleftharpoons (P \rightarrowtriangle Q) \land (\neg Q \rightarrowtriangle \neg P)$;
- $P \Leftrightarrow Q \rightleftharpoons (P \Rightarrow Q) \land (Q \Rightarrow P)$.

4.9 Theorem.
- $P, P \rightarrowtriangle Q \vdash Q$;
- if $\Gamma, P \vdash Q$, then $\Gamma \vdash P \rightarrowtriangle Q$;
- $P, P \Rightarrow Q \vdash Q$;
- $\neg Q, P \Rightarrow Q \vdash \neg P$.

4.10 Theorem.
- $\vdash \forall x\, (P \rightarrowtriangle Q) \Leftrightarrow \Upsilon x\, Q \mathrel{)\!\!=} \Upsilon x\, P$;
- $\vdash \forall x\, (P \leftrightarrowtriangle Q) \Leftrightarrow \Upsilon x\, Q = \Upsilon x\, P$.

4.11 Theorem (Replacement Rule for Comprising)**.**

- If $\begin{cases} u \text{ has only top occurrences in } P, \\ u \text{ is out of the scope in } P \text{ of any variable free in } \Gamma \text{ and in } \{t, t'\}, \end{cases}$

  then $\dfrac{\Gamma \vdash t \;)\!\!= t'}{\Gamma, P(u\|t) \vdash P(u\|t')}$.

4.12 Example. In the above theorem, the condition that u has only top occurrences in P is essential. Consider I a simple interpretation having $\mathbb{N}$ as its domain, which assigns to "<" its traditional meaning. Then

$$\Upsilon x \left(\Upsilon x \,(x = 2 \vee x = 3) < 3\right) < 2$$

and

$$\Upsilon x \,(x = 2 \vee x = 3) \;)\!\!= \Upsilon x \,(x = 2)$$

are true, but "$\Upsilon x \left(\Upsilon x \,(x = 2) < 3\right) < 2$" is false, according to I.

4.13 Theorem (Replacement Rule for Equivalence)**.**

- If $\begin{cases} \Gamma \vdash P \Leftrightarrow P', \\ Q \text{ is out of the scope in } R \text{ of any variable free in } \Gamma \text{ and in } \{P, P'\}, \end{cases}$

  then $\Gamma \vdash R(Q\|P) \Leftrightarrow R(Q\|P')$.

- If $\begin{cases} \Gamma \vdash P \Leftrightarrow P', \\ Q \text{ is out of the scope in } t \text{ of any variable free in } \Gamma \text{ and in } \{P, P'\}, \end{cases}$

  then $\Gamma \vdash t(Q\|P) = t(Q\|P')$.

4.14 Theorem (Replacement Rule for Equality)**.**

- If $\begin{cases} \Gamma \vdash t = t', \\ u \text{ is out of the scope in } P \text{ of any variable free in } \Gamma \text{ and in } \{t, t'\}, \end{cases}$

  then $\Gamma \vdash P(u\|t) \Leftrightarrow P(u\|t')$.

- If $\begin{cases} \Gamma \vdash t = t', \\ u \text{ is out of the scope in } v \text{ of any variable free in } \Gamma \text{ and in } \{t, t'\}, \end{cases}$

  then $\Gamma \vdash v(u\|t) = v(u\|t')$.

4.15 Theorem (∀-Elimination Rule for general terms)**.**

- If x has at most one free occurrence in P, then $\forall x\, P \vdash P(x|t)$.

4.16 Theorem (∃-Introduction Rule for general terms)**.**

- If x has at most one free occurrence in P, then $\mathrm{ex}(t), P(x|t) \vdash \exists x\, P$.

4.17 Theorem (Congruent Descriptions Rule)**.**

- If y is not free in P, then $\vdash \Upsilon x\, P = \Upsilon y\, P(x|y)$.

4.18 Theorem (Context Rules)**.**

- If x is top in Q,

  then $\begin{cases} * \;\; Q(x|\Upsilon x\, P) \vdash \forall x\, (P \multimap Q); \\ * \;\; \text{if } x \text{ has exactly one free occurrence in } Q, \\ \quad\; \text{then } \exists x\, P, \forall x\, (P \multimap Q) \vdash Q(x|\Upsilon x\, P). \end{cases}$

4.19 Corollary.

- If $\begin{cases} x \text{ is top in } Q, \\ x \text{ has exactly only one free occurrence in } Q, \\ Q \text{ is a basic atomic formula or a negation of a basic atomic formula,} \end{cases}$

  then $\begin{cases} Q(x|\Upsilon x\, P) \vdash \forall x\, (P \multimap Q); \\ \forall x\, (P \multimap Q) \vdash Q(x|\Upsilon x\, P). \end{cases}$

5 Elimination of Descriptions

In this section it is provided a translation from LAR to *Classical Equational Logic* ($P \longmapsto P_S$), in which all occurrences of "Υ"are eliminated and the remaining occurrences of "$\models$"can be interpreted as the equality sign.

5.1 Definition.
The following clauses specify the functions $P \longmapsto P_S$ and $P \longmapsto P_N$:
- if P has no occurrence of "Υ", then $P_S = P_N = P$;
- if P has one the forms $R(x|\Upsilon x\, Q)$ or $R(y|\Upsilon x\, Q)$,

  whereon $\begin{cases} * \;\; R \text{ is a basic atomic formula,} \\ * \;\; \text{“}\Upsilon x\, Q\text{” is the first occurrence, from left to right,} \\ \quad\; \text{of a description in } P, \end{cases}$

  then

  $*$ if $\begin{cases} P \text{ is of the first form,} \\ x \text{ is top in } R, \\ x \text{ has only one free occurrence in } R, \end{cases}$ then $\begin{cases} P_S = \forall x\, (Q_S \to R_S), \\ P_N = \exists x\, (Q_S \wedge R_N); \end{cases}$

  $*$ if $\begin{cases} P \text{ is of the second form} \\ y \text{ is the first variable such that} \end{cases}$ $\begin{cases} y \text{ is top in } R, \\ y \text{ has only one free occurrence in } R, \\ y \text{ is free in } Q, \end{cases}$

  then $\begin{cases} P_S = \forall y\, \big(Q(x|y)_S \to R_S\big), \\ P_N = \exists y\, \big(Q(x|y)_S \wedge R_N\big); \end{cases}$

- if $\begin{cases} \text{t is a non pure term,} \\ \text{x is the first variable that is not free in t,t'}, \end{cases}$
  then $\big(t' \models t\big)_S = \big(t' \models t\big)_N = \forall x \Big(\big(t \models x\big)_S \to \big(t' \models x\big)_S\Big)$;
- if $\begin{cases} t \text{ is a pure term,} \\ x_1, \ldots, x_n \text{ are the first } n \text{ variables that are not free in } t_1, \ldots, t_n, t, \end{cases}$
  then
  $$\big(f(t_1, \ldots, t_n) \models t\big)_S = \big(f(t_1, \ldots, t_n) \models t\big)_N =$$
  $$\exists x_1 \ldots \exists x_n \Big(\big(t_1 \models x_1\big)_S \wedge \ldots \wedge \big(t_n \models x_n\big)_S \wedge \big(t \models f(x_1, \ldots, x_n)\big)_S\Big);$$
- if t is a pure term, then $\big(\Upsilon x\, P \models t\big)_S = \big(\Upsilon x\, P \models t\big)_N = P_S(x|t)$;
- $(\neg P)_S = \neg P_N$;
- $(\neg P)_N = \neg P_S$;
- $(P \to Q)_S = P_N \to Q_S$;
- $(P \to Q)_N = P_S \to Q_N$;
- $(P \wedge Q)_S = P_S \wedge Q_S$;
- $(P \wedge Q)_N = P_N \wedge Q_N$;
- $(P \vee Q)_S = P_S \vee Q_S$;
- $(P \vee Q)_N = P_N \vee Q_N$;
- $(\forall x\, P)_S = \forall x\, P_S$;
- $(\forall x\, P)_N = \forall x\, P_N$;
- $(\exists x\, P)_S = \exists x\, P_S$;
- $(\exists x\, P)_N = \exists x\, P_N$.

5.2 Theorem.
- $\vdash P \Leftrightarrow P_S$;
- if P is a pure formula, then $\begin{cases} \vdash P \Leftrightarrow P_S; \\ \vdash P \Leftrightarrow P_N. \end{cases}$

5.3 Corollary (Correctness and Completeness)**.**
For LAR, $\Gamma \vdash P$ if, and only if, $\Gamma \models P$.

5.4 Corollary.
- $\vdash (P \to Q) \Leftrightarrow (P_N \to Q_S)$;
- $\vdash (P \leftrightarrow Q) \Leftrightarrow ((P_N \to Q_S) \wedge (Q_N \to P_S))$;
- $\vdash (P \rightharpoonup Q) \Leftrightarrow (P_S \to Q_S)$;
- $\vdash (P \Leftrightarrow Q) \Leftrightarrow (P_S \leftrightarrow Q_S)$;
- $\vdash (P \Rightarrow Q) \Leftrightarrow (P_S \to Q_S) \wedge (P_N \to Q_N)$;
- $\vdash (P \Leftrightarrow Q) \Leftrightarrow (P_S \leftrightarrow Q_S) \wedge (P_N \leftrightarrow Q_N)$.

5.5 Corollary (Modus Ponens (rewritten))**.** $P_N, P \to Q \vdash Q$.

5.6 Corollary (Deduction Rule (rewritten))**.** $\dfrac{\Gamma, P_N \vdash Q}{\Gamma \vdash P \to Q}$.

5.7 Corollary (Non Contradiction Rule (rewritten))**.**

- $\dfrac{\Gamma, P \vdash Q \qquad \Gamma, P \vdash \neg Q_S}{\Gamma \vdash \neg P_S}$;
- $\dfrac{\Gamma, P \vdash Q_N \qquad \Gamma, P \vdash \neg Q}{\Gamma \vdash \neg P_S}$;
- $\dfrac{\Gamma, P_N \vdash Q \qquad \Gamma, P_N \vdash \neg Q_S}{\Gamma \vdash \neg P}$;
- $\dfrac{\Gamma, P_N \vdash Q_N \qquad \Gamma, P_N \vdash \neg Q}{\Gamma \vdash \neg P}$.

5.8 Definition. Let P be a basic formula, $\Upsilon x_1\, P_1, \ldots, \Upsilon x_n\, P_n$ be all top occurrences of descriptions in P. For each $i = 1, \ldots, n$, let p_i be the number of variables free in $\Upsilon x_i\, P_i$ such that $\Upsilon x_i P_i$ is in their scope in P, and let $y_1^i, \ldots, y_{p_i}^i$ be these variables in alphabetical order. The following formulas are specified from P:

- $\mathrm{vac}(P) \rightleftharpoons \bigwedge\limits_{i=1}^{n} Q_i$, whereon $Q_i = \forall y_1^i \ldots \forall y_{p_i}^i\ \mathrm{vac}(\Upsilon x_i P_i)$;
- $\mathrm{ex}(P) \rightleftharpoons \bigwedge\limits_{i=1}^{n} R_i$, whereon $R_i = \forall y_1^i \ldots \forall y_{p_i}^i\ \mathrm{ex}(\Upsilon x_i\, P_i)$;
- $\mathrm{un}(P) \rightleftharpoons \bigwedge\limits_{i=1}^{n} S_i$, whereon $S_i = \forall y_1^i \ldots \forall y_{p_i}^i\ \mathrm{un}(\Upsilon x_i P_i)$;
- $\mathrm{amb}(P) \rightleftharpoons \bigwedge\limits_{i=1}^{n} T_i$, whereon $T_i = \forall y_1^i \ldots \forall y_{p_i}^i\ \mathrm{amb}(\Upsilon x_i P_i)$;
- "vac(P)" is read "all top descriptions in P are vacuous", or simply "P is vacuous";
- "ex(P)" is read "all top descriptions in P are existential", or simply "P is existential";
- "un(P)" is read "all top descriptions in P are univocal", or simply "P is univocal";
- "amb(P)" is read "all top descriptions in P are ambiguous", or simply "P is ambiguous".

The next result provides simpler equivalent forms for P_S and P_N, given a formula P satisfying some reasonable restrictions.

5.9 Theorem (Easy Elimination of Descriptions)**.**
Let P be a basic formula of the form $Q(x_1, \ldots, x_n | \Upsilon x_1\, P_1, \ldots, \Upsilon x_n\, P_n)$, obtained from Q by instantiating $x_1, \ldots, x_n$ simultaneously by $\Upsilon x_1\, P_1, \ldots, \Upsilon x_n\, P_n$, satisfying the following conditions:

- Q is a pure formula;
- $x_1, \ldots, x_n$ are distinct variables top in Q;

- for $i \neq j$, no x_i is free in some P_j;
- each x_i has only one free occurrence in Q;
- each description $\Upsilon x_i P_i$ is not in the scope in P of some variable free in this description;
- for $i = 1, \dots, n$, $\Gamma \vdash \exists x_i P_i$.

Then the following propositions are valid:

- $\Gamma \vdash P_S \Leftrightarrow \forall x_1 \dots \forall x_n \big((P_1)_S \wedge \dots \wedge (P_n)_S \to Q\big)$;
- $\Gamma \vdash P_S \Leftrightarrow \forall x_1 \dots \forall x_n \,(P_1 \wedge \dots \wedge P_n \twoheadrightarrow Q)$;
- $\Gamma \vdash P_N \Leftrightarrow \exists x_1 \dots \exists x_n \big((P_1)_S \wedge \dots \wedge (P_n)_S \wedge Q\big)$.

For the following five corollaries, consider that P is a basic formula.

5.10 Corollary (Uniqueness Rule)**.** $\begin{cases} \mathrm{un}(P) \vdash P \Leftrightarrow P_S; \\ \mathrm{un}(P) \vdash P \Leftrightarrow P_N. \end{cases}$

5.11 Corollary (Existence Rule)**.** $\mathrm{ex}(P), P_S \vdash P_N$.

5.12 Corollary (Non Ambiguity Rule)**.** $\neg\,\mathrm{amb}(P), P_N \vdash P_S$.

5.13 Corollary (Modus Ponens (clean version))**.** $\mathrm{ex}(P), P, P \to Q \vdash Q$.

5.14 Corollary (Deduction Rule (clean version))**.** $\dfrac{\Gamma, \neg\,\mathrm{amb}(P), P \vdash Q}{\Gamma \vdash P \to Q}$.

5.15 Definition.

- $\sim P \rightleftharpoons \neg(P_S)$;
- $P^\circ \rightleftharpoons \sim(P \wedge \neg P)$;
- $P^* \rightleftharpoons P \vee \neg P$.

Observe, by a simple reasoning, that $\vdash P^* \Leftrightarrow (P \to P)$.

According to the following lemma, the sign "$\sim$" works as classical negation.

5.16 Lemma. $\dfrac{\Gamma, \sim P \vdash Q \qquad \Gamma, \sim P \vdash \sim Q}{\Gamma \vdash P}$.

5.17 Theorem (Non Contradiction Rule (first clean version))**.**

- $\dfrac{\Gamma \vdash P^* \qquad \Gamma \vdash Q^\circ \qquad \Gamma, P \vdash Q \qquad \Gamma, P \vdash \neg Q}{\Gamma \vdash \neg P}$.

5.18 Lemma. If P is a basic formula, then

- $\mathrm{ex}(P) \vdash P^\circ$;
- $\neg\,\mathrm{amb}(P) \vdash P^*$.

5.19 Corollary (Non Contradiction Rule (second clean version))**.**

- If P is a basic formula,
  then $\dfrac{\Gamma \vdash \neg\,\mathrm{amb}(P) \qquad \Gamma \vdash \mathrm{ex}(Q) \qquad \Gamma, P \vdash Q \qquad \Gamma, P \vdash \neg Q}{\Gamma \vdash \neg P}$.

Sometimes it is not possible to prove "$P \to Q$" in some environment by using some of the results given above. Taking into account this possibility, another version of a deduction rule is provided below.

5.20 Theorem (Deduction Rule (practical version))**.**

- Let P be a basic formula of the form $Q(x_1, \ldots, x_n | \Upsilon x_1 P_1, \ldots, \Upsilon x_n\, P_n)$, obtained from Q satisfying the same conditions of theorem 5.9.
  If $\begin{cases} \Gamma, (P_1)_S \wedge \ldots \wedge (P_n)_S \wedge Q \vdash R, \\ x_1, \ldots, x_n \text{ are not free in } \Gamma \cup \{R\}, \end{cases}$ then $\Gamma \vdash P \to R$.

6 A comparison between the qualifier "Υ" and other ones

There are some approaches for qualifiers in scientific literature. Maybe the most known are Russell's and Hilbert's [11, 12, 10, 6].

In Russell [11] it is described a version of definite article. Russell introduces the symbol "ι" in a contextual way:

- $Q(x|\iota x\, P) \rightleftharpoons \exists x\, (P \wedge Q \wedge \forall y\, (P(x|y) \to y = x))$.

Russell's approach doesn't consider "$\iota x\, P$" as a name at all, but the whole expression "$Q(x|\iota x\, P)$" only as an abbreviation. Although it can be convenient for doing mathematics, this approach introduces a sign with no recognized linguistic status, that is, expressions like "$\iota x\, P$" don't have any linguistic value by their own, although they are in fact used.

Other approaches, like ours, consider descriptions as "$\iota x\, P$", "$\varepsilon x\, P$" and "$\Upsilon x\, P$" as names. According to them, "$\iota x\, P$" denotes the only object x satisfying P, whereas "$\varepsilon x\, P$" denotes a fixed object x satisfying P, chosen from the collection of all objects satisfying P.

The main problem for "ι" is what to ascribe to "$\iota x\, P$" when there is no object or more than one object x satisfying P, whereas the analogous problem for "ε" is what to assign to "$\varepsilon x\, P$" when there is no object x satisfying P. All known approaches to these situations assign to these descriptions some object of the domain, but their main sin is their lack of uniformity in dealing with this kind of circumstance.

Our approach, on the contrary, has no special clause for dealing with the circumstance in which there is no object x satisfying P. "$\Upsilon x\, P$" is associated, according to the semantics of LAR, to the collection of all objects

x satisfying P. It is implicit in LAR semantics that "$\Upsilon x\, P$" is an ambiguous name when there is more than one object x satisfying P; there is no choice in this case (as it is done for the "ε" approach), that is, "$\Upsilon x\, P$" is a name for each object x satisfying P, with no preference or choice by a particular object over another one. If there is only one such object x satisfying P, "$\Upsilon x\, P$" becomes a definite name for this object. Finally, if there is no object x satisfying P, then "$\Upsilon x\, P$" is a name for nothing, that is, it is a *vacuous name*.

Given a term t containing descriptions, the traditional approaches don't inform us easily if this term is a vacuous name or not. It can be true, in some context, for example, that $t = \emptyset$, but we don't know, only by examining this expression, if "$\emptyset$" was obtained as a result of some reasoning or computation, or if "$\emptyset$" is being used as a label for a vacuous name. Our approach instead has a direct way for saying that a name is vacuous, simply by writing "vac(t)". It is equally easy to say that this name is *existential*, *ambiguous*, or *univocal*, as it was already shown above.

It is also possible to define in LAR a kind of definite article, as it is shown below:

- $\iota x\, P \rightleftharpoons \Upsilon x\, (P \wedge \forall y\, (P(x|y) \rightarrow y = x))$.

For this "ι", according to the definition above, "$\iota x\, P$" is a name for nothing, if there is no object x or if there is more than one object x satisfying P.

There is another important failure related to the "ε" approach, which will be shown in the following example.

6.1 Example. We know that, in category theory, two objects a, b of the same category can have more than one product, but we represent a product of a and b by "$a \times b$". We also know that $a \times b$ and $b \times a$ are isomorphic objects, which we denote by "$a \times b \approx b \times a$", there is, each product of a and b is isomorphic to each product of b and a. If we define categorial product by using "ε" , then, as the collections of all products of a and b and of all products of b and a are the same, then the expression "$a \times b \approx b \times a$" means only that, for each x being a product of a and b (or of b and a), $x \approx x$, which is very poor for the original intended meaning. If we use instead "Υ"for defining categorial product, then the expression "$a \times b \approx b \times a$" has the intended meaning.

7 Conclusions

Our way in doing semantics presents a new paradigm, by dealing explicitly with ambiguity and vacuity, otherwise to most semantics. Even modal logics, with non rigid designators semantics, don't deal essentially with ambiguity, because in the same world there are no variations of reference.

We believe traditional mathematics lacks a logical basis managing ambiguity and vacuity. They appear in mathematics in many places, from set theory to mathematical analysis and number systems. Many propositions of theorems could be very simplified, and maybe this expansion of language could open fresh roads for new discoveries.

In natural language most phrases use ambiguous names for referencing objects, so Υ-descriptions appear to be a natural way for modelling these situations.

We don't claim that LAR is a kind of "final" or "perfect" logic for dealing with ambiguity or with the problems just pointed out, but that it is a new departure point, from which it is necessary a possibly long path for reaching something very useful. While conceiving this logic, we have realized that there are also *existential descriptions*, and that descriptions, being universal or existential, can be *linked* or not. For modelling a reasonable logic taking into account these new ideas, expressing in a natural way deep intuitions, all rush is enemy of perfection.

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