1 Kinship Network Decomposition

Given a kinship network $G = (V, A, E)$ (V is the set of vertices, A is the set of arcs and E is the set of edges), a *Kinship Network Decomposition* (KND) is a partition of V in $V_1, V_2, \ldots, V_k$ such that k is minimum and, for all $i = 1, 2, \ldots k$:

- there is no edge inside V_i; and
- the directed graph induced by V_i is (weakly) connected.

The *kinship network decomposition* problem asks for finding a KND from a given kinship network.

The motivation to find such decomposition comes from the Anthropology Area. For Krahô people [1], it is not good to marry consanguineous relatives. So, an individual looks for a partner outside its own relative group [2],[3]. We are using kinship networks to represent individuals and connection between individuals: a vertex represents an individual, an arc $u \to v$ represents a parental connection and an edge $u - v$ represents an affinity connection (a marriage). A KND would show which families each individual could marry with.

1.1 O que pode ser feito?

We are interested in finding a KND from kinship networks. We can work on empirical network (Krahô) which has 1031 vertices and 2779 edges plus arcs. We can try to develop exact or approximate algorithms for the problem. We also can try to find out the difficulty of the problem when the graph induced by the edges of the input is bipartite. We also can try to find out MIP for the problem.

1.2 Estado da arte

The problem is NP-hard when we have no constraints over the graph induced by the edges of the input kinship network. We have a reduction from the chromatic number problem. We also have an exact algorithm whose time complexity is pretty bad.

2 Results in WoPOCA

2.1 Complexity in kinship network constrained)

Lemma 1 *Let D be a kinship network. The KND of D is NP-hard even if the graph induced by E is bipartite.*

Proof 1 *The reduction is from minimum coloring problem in graph. Let $G = (V, E)$ be a graph. Let us construct a graph $D = ((P, Q), A', E')$ such that G is k-chromatic if and only if there exist a k KND in D.*

Let $v_1, \ldots, v_n$ the vertices of G.

Let $P = \{v_1^P, \ldots, v_n^P\}$, $Q = \{v_1^Q, \ldots, v_n^Q\}$, $A' = \{v_i^P \to v_i^Q : v_i \in V\} \cup \{v_i^P \to v_j^P : v_i, v_j \in V, i < j\}$ and $E' = \{v_i^P - v_j^Q : v_i - v_j \in E\}$, .

Let us show that if there exists a k coloring in G then there exists a k KND in D. Let $C_1, \ldots, C_k$ a k coloring in G. Let $C'_r = \{v_i^P : v_i \in C_r\} \cup \{v_i^Q : v_i \in C_r\}$. Note that C'_r is connected since there exist arc $v_i^P \to v_i^Q$ in D and any set of vertices in P is connected. Also note that C'_r is edge free since all pair of vertices v_i^P, v_j^Q in C'_r there exist a pair v_i, v_j in C_r which is an independet set in G.

Let show now that if there exists a minimum k KND in D then there exists a k coloring in G. Let $C'_1, \ldots, C'_k$ a minimum k KND in D. Since the set Q is independent any C'_r contains vertices v_i^P and v_i^Q or only v_i^Q. Suppose v_i^Q alone in C'_r and v_i^P in C'_s ($s \neq r$). If the size of $|C'_s| = 1$ we could elapse C'_s and C'_r creating a better decomposition. So $|C'_s| > 1$. Then we can change the set that v_i^P belongs from C'_s to C'_r and allways is possible put togethe v_i^P and v_i^Q in a same set C'_r. In this way we can create a k coloring $C_1, \ldots, C_k$ where v_i is in C_r if v_i^P and v_i^Q are in C'_r. Note that C_r will be independent since C'_r has allways v_i^P and v_i^Q and it is edge free.

2.2 Formulations

Flow Formulation

Vertex s is a new vertex. There are arcs from s to every other vertex v. n is the number of vertices in the input kinship network. Every original arc $u \to v$ is seen as parallel arcs $u \to v$ and $v \to u$.

Binary Variables

$$x_{vi} = \begin{cases} 1 & \text{if vertex } v \text{ is in component } i, \\ 0 & \text{otherwise.} \end{cases}$$

$$y_{uv} = \begin{cases} 1 & \text{if arc } u \to v \text{ is choosen to garantee component connected,} \\ 0 & \text{otherwise.} \end{cases}$$

Integer Variables

$$f_{uv} \geq 0 \text{ for all arc } u \to v.$$

$$\begin{array}{rcll} \text{Minimize} \sum_{uv \in N^+(s)} y_{uv} & & & \\ \sum_{u \in N^-(v)} f_{uv} - \sum_{u \in N^+(v)} f_{vu} & = & 1 & \text{for all vertex } v \\ \sum_{v \in V} f_{sv} & = & n & \\ y_{sv} & \leq & x_{vv} & \text{for all } v \\ x_{ui} + y_{uv} - 1 & \leq & x_{vi} & \text{for all arc } u \to v\text{, for all component } i \\ f_{uv} & \leq & n.y_{uv} & \text{for all arc } u \to v \\ \sum_{u \in N^-(v)} y_{uv} & = & 1 & \text{for all vertex } v \\ x_{ui} + x_{vi} & \leq & 1 & \text{for all edge } u - v\text{, for all component } i \end{array}$$

Cut Formulation

Binary Variables

$$x_{vi} = \begin{cases} 1 & \text{if vertex } v \text{ is in component } i, \\ 0 & \text{otherwise.} \end{cases}$$

$$w_i = \begin{cases} 1 & \text{if component } i \text{ exists,} \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{array}{rcll} \text{Minimize} \sum_{i=1}^{n} w_i & & & \\ \sum_{i=1}^{n} x_{vi} & = & 1 & \text{for all vertex } v \\ x_{ui} + x_{vi} & \leq & 1 & \text{for all edge } u - v\text{, for all component } i \\ x_{vi} & \leq & w_i & \text{for all component } i\text{, for all vertex } v \\ x_{vi} + x_{ui} - \sum_{z \in S} x_{zi} & \leq & 1 & \text{for all pair } u, v \text{ which is not an arc,} \\ & & & \text{for all vertex-separator}(u, v) \end{array}$$

References

[1] J. C. Melatti. Povos Indígenas no Brasil: Krahô. `https://pib.socioambiental.org/en/Povo:Krahô`, 2018. Online; accessed in ago, 21, 2018.

[2] C. M. d. O. Paulino. A rede Mehĩ: em busca de estruturas de troca e parentesco Kraho. Master's thesis, Univ. of São Paulo, 2016.

[3] C. M. d. O. Paulino and Á. J. P. Franco. The Kraho Empiric Network: an exercise in modelling. In *Proc. 18th IUAS World Congress*, Florianópolis, Brazil, To appear.